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Blocking Probability in a Switching Center With Arbitrary Routing Policy

By B. GOPINATH,* J.-M. GARCIA,† and P. VARAIYA‡

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We model the arrival of calls at a switch where they are assigned to any one of the available idle outgoing links. A call is blocked if all links are busy. It may be lost after assignment to an idle link with a probability that depends on the link. For the case of Poisson arrivals and exponential holding times, all show that the distribution of passage time to the blocked state is independent of the assignment policy. As a consequence, the blocking probability and the law of the overflow traffic are also independent of the assignment policy.

I. INTRODUCTION

Telephone calls arrive at a switching center in a Poisson stream of rate λ. When a call arrives, the switch, in accordance with a prespecified policy, assigns the call to one of the idle outgoing links. If the call is assigned to the idle link i, there is a probability $1 - \epsilon_i$ that the call is unsuccessful, and probability $\epsilon_i > 0$ that it is successful. An unsuccessful call is immediately lost and link i remains idle. If the call is successful, link i immediately becomes busy and remains in that state for a holding or conversation time, which is an exponentially distributed random time with mean $1/\mu$. At the end of this holding time link i returns to the idle state again.

The preceding paragraph describes a model that approximates the behavior of a single node in a telephone network. In an actual network,

* AT&T Bell Laboratories; present affiliation Bell Communications Research, Inc. † Toulouse, France. ‡ University of California, Berkeley, CA.

for a call to be placed successfully a circuit or path of idle links must be established between the call's originating node and the destination node. The node where the call arrives assigns it to one of its idle links, say, link i. As the call proceeds from node to node towards its destination, it may encounter a node all of whose outgoing links are busy, in which case the call is lost. The probability that this may happen is $1 - \epsilon_i$. This probability depends on the entire path that the call follows. It is assumed here, however, that the probability depends only on the initial link i. Such a link-by-link analysis is a common approximation in traffic theory.[1]

Suppose the model has n outgoing links indexed $i \in \Omega := \{1, 2, \cdots, n\}$. The state of the links or of the switching node is the subset $I \subset \Omega$ of idle links. So the state space is 2^Ω and there are 2^n states. The state where there is no idle link, namely state ϕ, is the blocked state. In a manner specified precisely in the next section, each switching policy u defines an irreducible Markov chain on this state space. Let $\tau_u(I)$ be the first time that, in equilibrium, the chain starting in state I reaches the blocked state ϕ, and let ρ_u be the time to return to ϕ after leaving ϕ.

In this paper we show the surprising fact that the distributions of $\tau_u(I)$ and ρ_u are independent of the policy u. Several consequences follow from this: First, the blocking probability $p_u(\phi)$, an important performance measure, is independent of u. Second, the process of blocked calls[4] has a law that does not depend on the policy u either. Hence, in designing a switching policy one need not worry about blocked calls since they cannot be affected by the policy anyway. The overflow process, which consists of both lost and blocked calls, does of course depend on the policy.

This paper is organized as follows. Section II presents the Markov chain description. Section III gives the calculation of the distribution of the passage time $\tau_u(I)$. An algorithm is given to evaluate these distributions. Section IV contains two examples. For the special case where there are only two possible values for the loss probabilities ϵ_i, the algorithm simplifies. For this case a formula for the blocking probability is conjectured. The formula extends the well-known Erlang formula and it has been verified for many examples. However, a proof of its correctness is not yet available.

II. THE MARKOV CHAIN DESCRIPTION

A policy u prescribes, for each state I, the probability with which an arriving call is assigned to an idle link $i \in I$. Thus u is specified by an array $u = \{u(I, i), i \in I \neq \phi\}$ such that $u(I, i) \geq 0$, $\sum_i u(I, i) = 1$.

Each u defines a transition rate matrix $\mathbf{R}_u = \{R(I, J), I \subset \Omega, J \subset \Omega\}$

whose nonzero elements are

$$\mathbf{R}_u(I, I - i) = \lambda_i u(I, i), \qquad i \in I$$

$$\mathbf{R}_u(I, I + j) = \mu, \qquad j \notin I$$

$$\mathbf{R}_u(I, I) = -\sum_{i \in I} \lambda_i u(I, i) - (n - |I|)\mu, \tag{1}$$

where $\lambda_i := \epsilon_i \lambda$, $I - i := I \backslash i$, $I + j := I \cup \{j\}$, and $|I|$ is the cardinality of I. The first expression in (1) gives the rate with which an idle link becomes busy, the second gives the rate with which a busy link becomes idle, the third gives the diagonal term $\mathbf{R}_u(I, I) = \sum_{J \neq I} \mathbf{R}_u(I, J)$. For later reference observe that the column of $\mathbf{R}_u$ corresponding to the blocked state has elements

$$\mathbf{R}_u(\phi, \phi) = -n\mu, \qquad \mathbf{R}_u(\{i\}, \phi) = \lambda_i,$$
$$\mathbf{R}_u(I, \phi) = 0 \quad \text{for} \quad |I| > 1. \tag{2}$$

Since each $\epsilon_i > 0$ by assumption, it follows that the chain defined by $\mathbf{R}_u$ is irreducible and possesses a unique invariant positive probability measure $\{p_u(I), I \subset \Omega\}$. In particular, $p_u(\phi)$ is the equilibrium blocking probability. It will be shown to be independent of u. This is surprising in view of the fact that, if the ϵ_i are all different, then for every $I \neq \phi$, $p_u(I)$ depends on u.

Suppose the chain starts in state $I \neq \phi$ at time 0 and let $\tau_u(I)$ denote the first time the chain reaches ϕ. The distribution of $\tau_u(I)$ can be obtained as follows. Let $\hat{\mathbf{R}}_u$ be the rate matrix obtained from $\mathbf{R}_u$ by making ϕ an absorbing state:

$$\hat{\mathbf{R}}_u(\phi, j) \equiv 0, \qquad \hat{\mathbf{R}}_u(I, J) \equiv \mathbf{R}_u(I, J), \qquad I \neq \phi. \tag{3}$$

Let $\mathbf{e}_J = \{\mathbf{e}_J(I), I \subset \Omega\}$ denote the (column) vector whose only nonzero component is $\mathbf{e}_J(J) = 1$. Let $p_t = \{p_t(J)\}$ be the solution of

$$p_t(J) = \sum_K p_t(K)\hat{\mathbf{R}}_u(K, J), \qquad p_0 = \mathbf{e}_I. \tag{4}$$

Then one has

$$\text{Prob}\{\tau_u(I) \leq t\} = p_t(\phi),$$

i.e., $p_t(\phi)$ is the cumulative distribution function of the first passage time $\tau_u(I)$. From (4) it follows that the Laplace transform $F_u(I, s)$ of $p_t(\phi)$ is given by

$$F_u(I, s) = \mathbf{e}_I'(s\mathscr{I} - \hat{\mathbf{R}}_u)^{-1}\mathbf{e}_\phi, \tag{5}$$

where $\mathscr{I}$ is the identity matrix.

The next section is devoted to the main result:

Theorem 1: $(s\mathscr{I} - \hat{\mathbf{R}}_u)^{-1}\mathbf{e}_\phi$ *does not depend on* u.

The theorem has several consequences. First, it shows that the distribution of the passage time $\tau_u(I)$ does not depend on u. Second, suppose the chain corresponding to $\mathbf{R}_u$ leaves state ϕ at time 0-. Then at time 0 it enters one of the states $\{1\}, \cdots, \{n\}$, each with probability $1/n$. Let ρ_u be the first time that the chain returns to ϕ. The Laplace transform $G_u(s)$ of $\text{Prob}\{\rho_u \le t\}$ is simply

$$G_u(s) = \frac{1}{n} \sum_{i=1}^{n} F_u(\{i\}, s),$$

which is independent of u. In particular, the expected value $\bar{\rho}$ of ρ_u is independent of u. The expected holding time in state ϕ is $1/n\mu$, and so the blocking probability

$$p_u(\phi) = \frac{1}{1 + n\mu\bar{\rho}}$$

is also independent of u. Finally, the process of blocked calls can be described by a sequence of independent intervals $S_1, T_1, S_2, T_2, \cdots$. Each S_i has the same distribution as $\rho_u(\phi)$, and during this interval there is no blocked call. Each T_i has the same distribution as the holding time in state ϕ, and during this interval all arriving calls are blocked. Thus the statistics of the blocked call process are not affected by u.

III. CALCULATION OF $(s\mathscr{I} - \mathbf{R}_u)^{-1}\mathbf{e}_\phi$

Throughout this section a fixed policy is considered and so the suffix u is dropped.

Using (1) through (3) write

$$\hat{\mathbf{R}} = A_\lambda + A_\mu + B,$$

with the nonzero elements of these matrices being

$$A_\lambda(I, I - i) = \lambda_i u(I, i), \qquad i \in I \quad \text{and} \quad I - i \ne \phi$$

$$A_\lambda(I, I) = - \sum_{i \in I} \lambda_i u(I, i), \qquad I \ne \phi$$

$$A_\mu(I, I + j) = \mu, \qquad j \notin I \quad \text{and} \quad I \ne \phi$$

$$A_\mu(I, I) = -(n - |I|)\mu, \qquad I \ne \phi$$

$$B(\{i\}, \phi) = R(\{i\}, \phi) = \lambda_i.$$

Then

$$A_\lambda(\phi,\ \cdot) \equiv A_\lambda(\cdot,\ \phi) \equiv A_\mu(\phi,\ \cdot) \equiv A_\mu(\cdot,\ \phi) \equiv 0, \tag{6}$$

and for any vector $x = \{x(I)\}$,

$$(A_\lambda x)(I) := \sum_j A_\lambda(I, J)x(J) = \sum_{\substack{i \in I \\ I-i \neq \phi}} \lambda_i u(I, i)x(I-i)$$

$$- \sum_{i \in I} \lambda_i u(I, i)x(I), \tag{7}$$

$$(A_\mu x)(I) = \mu \sum_{j \notin I} x(I+j) - (n - |I|)\mu x(I), \quad I \neq \phi$$

$$= 0, \qquad I = \phi. \tag{8}$$

First one calculates $\hat{\mathbf{R}}^d \mathbf{e}_\phi$, $d \geq 0$.

From (6)

$$\mathbf{v}_1 := \hat{\mathbf{R}}\mathbf{e}_\phi = B\mathbf{e}_\phi \tag{9}$$

so its only nonzero components are $\mathbf{v}_1(\{i\}) = \lambda_i$. In particular, since $\mathbf{v}_1(\phi) = 0$, it follows that

$$\hat{\mathbf{R}}^d \mathbf{e}_\phi = (A_\lambda + A_\mu)^{d-1}\mathbf{v}_1, \qquad d \geq 1. \tag{10}$$

The appendix introduces vectors $\mathbf{w}_d$ and $\mathbf{z}_d$, $d \geq 1$. [See eqs. (16) through (19) and (24) through (26).] Define vectors $\mathbf{v}_d$ by

$$\mathbf{v}_d(I) = (-1)^{|I| \cdot d-1}\mathbf{w}_d(I), \qquad 1 \leq d \leq n+1. \tag{11}$$

It can be checked that $\mathbf{v}_1$ given by (9) and (11) are the same.

Next one evaluates $A_\lambda \mathbf{v}_d$.

Lemma 1:

$$\mathbf{v}_{d+1} = A_\lambda \mathbf{v}_d, \qquad 1 \leq d \leq n \tag{12}$$

$$\mathbf{v}_{n+1} = -\sum_{r=1}^{n} \mathbf{z}_r(\Omega)\mathbf{v}_{n+1-r}. \tag{13}$$

Proof: Since $\mathbf{w}_d(\phi) = \mathbf{v}_d(\phi) = 0$ and since $A_\lambda(\cdot,\ \phi) \equiv 0$, therefore, trivially,

$$\mathbf{v}_{d+1}(\phi) = (A_\lambda \mathbf{v}_d)(\phi) = 0.$$

Now suppose $I = \{i\}$. From (7) we get

$$\begin{aligned}(A_\lambda \mathbf{v}_d)(\{i\}) &= -\lambda_i \mathbf{v}_d(\{i\}) \\ &= -\lambda_i(-1)^d \mathbf{w}_d(\{i\}) \\ &= (-1)^{d+1}\mathbf{w}_{d+1}(\{i\}), \quad \text{by Lemma 3,} \\ &= \mathbf{v}_{d+1}(\{i\}).\end{aligned}$$

Next suppose $|I| \geq 2$. From (7)

$$(A_\lambda \mathbf{v}_d)(I) = \sum_{i \in I} \lambda_i u(I, i)\mathbf{v}_d(I - i) - \sum_{i \in I} \lambda_i u(I, i)\mathbf{v}_d(I)$$

$$= \sum_{i \in I} u(I, i)(-1)^{|I|+d}[\lambda_i \mathbf{w}_d(I) + \lambda_i \mathbf{w}_d(I - i)]$$

$$= (-1)^{|I|+d} \left[\sum_{i \in I} u(I, i) \right] \mathbf{w}_{d+1}(I), \quad \text{by Lemma 3,}$$

$$= (-1)^{|I|+d} \mathbf{w}_{d+1}(I) = \mathbf{v}_{d+1}(I).$$

Finally,

$$\mathbf{v}_{n+1}(I) = (-1)^{|I|+n} \mathbf{w}_{n+1}(I)$$

$$= -(-1)^{|I|+n} \sum_{r=1}^{n} (-1)^r \mathbf{z}_r(\Omega)\mathbf{w}_{n+1-r}(I), \quad \text{by Lemma 7,}$$

$$= -\sum_{r=1}^{n} \mathbf{z}_r(\Omega)\mathbf{v}_{n+1-r}(I). \qquad \square$$

Next one evaluates $A_\mu \mathbf{v}_d$. Define

$$q_r := \sum_{i \in \Omega} \lambda_i^r.$$

Lemma 2: $$A_\mu \mathbf{v}_d = -\mu \sum_{r=1}^{d-1} (-1)^r q_r \mathbf{v}_{d-r} - \mu(n - d)\mathbf{v}_d, \qquad d \geq 1. \tag{14}$$

Proof: From (7), and for $I \neq \phi$,

$$(A_\mu \mathbf{v}_d)(I) = \mu \sum_{j \notin I} \mathbf{v}_d(I + j) - (n - |I|)\mu \mathbf{v}_d(i)$$

$$= \mu(-1)^{|I|+d} \left[\sum_{j \notin I} \mathbf{w}_d(I + j) + (n - |I|)\mathbf{w}_d(I) \right]$$

$$= \mu(-1)^{|I|+d} \left[\sum_{j \notin I} \sum_{r=1}^{d-1} \lambda_j^r \mathbf{w}_{d-r}(I) + \sum_{r=1}^{d-1} \sum_{j \in I} \lambda_j^r \mathbf{w}_{d-r}(I) \right.$$

$$\left. + (n - d)\mathbf{w}_d(I) \right], \quad \text{by Lemmas 4 and 6,}$$

$$= \mu(-1)^{|I|+d} \left[\sum_{r=1}^{d-1} q_r \mathbf{w}_{d-r}(I) + (n - d)\mathbf{w}_d(I) \right]$$

$$= -\left[\sum_{r=1}^{d-1} (-1)^r q_r \mathbf{v}_{d-r}(I) + (n - d)\mathbf{v}_d(I) \right]. \qquad \square$$

These two lemmas yield the next corollary.

Corollary 1: For each $d \geq 1$, there exist coefficients $\alpha_{d1}, \cdots, \alpha_{dd\wedge n}$ not depending on u such that

$$\hat{\mathbf{R}}^d \mathbf{e}_\phi = \sum_{e=1}^{d\wedge n} \alpha_{de} \mathbf{v}_e,$$

where $d \wedge n := \min(d, n)$.

Proof: This is apparent from eqs. (10) through (14) and the fact that the vectors $\mathbf{w}_d$, $\mathbf{z}_d$, and q_r do not depend on u.

Corollary 1 implies the existence of coefficients $\alpha_0(s), \cdots, \alpha_n(s)$ such that

$$(s\mathscr{I} - \hat{\mathbf{R}})^{-1}\mathbf{e}_\phi = \alpha_0 \mathbf{e}_\phi + \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n.$$

These coefficients can be readily calculated. Multiplying both sides by $(s\mathscr{I} - \hat{\mathbf{R}})$,

$$\begin{aligned}
\mathbf{e}_\phi &= \alpha_0(s\mathscr{I} - \hat{\mathbf{R}})\mathbf{e}_\phi + \sum_1^n \alpha_d(s\mathscr{I} - \hat{\mathbf{R}})\mathbf{v}_d \\
&= \alpha_0 s\mathbf{e}_\phi - \alpha_0\mathbf{v}_1 + \sum_1^n \alpha_d s\mathbf{v}_d - \sum_1^n \alpha_d A_\lambda \mathbf{v}_d - \sum_1^n \alpha_d A_\mu \mathbf{v}_d \\
&= \alpha_0 s\mathbf{e}_\phi + \sum_1^n (\alpha_d s - \alpha_{d-1})\mathbf{v}_d - \alpha_n \sum_1^n \mathbf{z}_{n+1-d}(\Omega)\mathbf{v}_d \\
&\quad + \mu \sum_1^n \alpha_d \sum_{r=1}^{d-1} (-1)^r q_r \mathbf{v}_{d-r} + \mu \sum_1^n \alpha_d(n-d)\mathbf{v}_d.
\end{aligned}$$

Rearranging terms and equating the coefficients of $\mathbf{e}_\phi, \mathbf{v}_1, \cdots, \mathbf{v}_n$ gives

$$\alpha_0 = s^{-1}$$

$$\alpha_{d-1} = [s + (n-d)\mu]\alpha_d + \mu \sum_{r=1}^{n-d} (-1)^r q_r \alpha_{d+r}$$
$$- \mathbf{z}_{n+1-d}(\Omega)\alpha_n, \qquad 1 \leq d \leq n.$$

The last n equations are in "triangular" form and can be solved recursively to yield

$$\alpha_{n-r} = Q_r(s)\alpha_n,$$

where $Q_r(s)$ is a monic polynomial in s of degree r. This gives

$$\alpha_0 = \frac{1}{s} = Q_n(s)\alpha_n.$$

Hence

$$(s\mathscr{I} - \hat{\mathbf{R}})^{-1}\mathbf{e}_\phi = \frac{1}{s}\,\mathbf{e}_\phi + \frac{1}{sQ_n(s)}\,[Q_{n-1}(s)\mathbf{v}_1 + \cdots + Q_1(s)\mathbf{v}_{n-1} + \mathbf{v}_n].$$

Since the right-hand side does not depend on u, this proves Theorem 1, announced in the previous section.

IV. EXAMPLES

Suppose the n links are grouped into two "trunks," trunk 1 with n_1 and trunk 2 with n_2 links, $n_1 + n_2 = n$. The loss probability of every link in trunk i is the same, namely, $1 - \epsilon_i$. In this special case one may construct a Markov chain with state (i_1, i_2), where $0 \le i_1 \le n_1$ and $0 \le i_2 \le n_2$ are the number of idle links in trunks 1 and 2, respectively. The number of states reduces to $(n_1 + 1)(n_2 + 1)$. (Of course, if links within the same trunk are distinguished, then 2^n states are needed as before.) This reduction carries over to the argument in the appendix, and so one can define the "generating" vectors $\mathbf{w}_d$, $\mathbf{v}_d$ with components indexed by the reduced state (i_1, i_2). The recursive definition of $\mathbf{w}_d$ can be explicitly solved to obtain

$$\mathbf{w}_d(i_1, i_2) = \sum_{k=i_1}^{d-i_2} B(k, i_1)B(d-k, i_2)\lambda_1^k\lambda_2^{d-k},$$

where

$$B(k, i) = \sum_{j=0}^{k-i} \binom{k-2-j}{i-j}.$$

Also, as before,

$$\mathbf{v}_d(i_1, i_2) = (-1)^{i_1+i_2+d-1}\mathbf{w}_d(i_1, i_2), \qquad 1 \le d \le n$$

$$\mathbf{v}_{n+1}(i_1, i_2) = -\sum_{r=1}^{n} \mathbf{z}_r(n_1, n_2)\mathbf{v}_{n+1-r}(i_1, i_2),$$

where

$$\mathbf{z}_d(i_1, i_2) = \sum_{k=0}^{d} C(k, i_1)C(d-k, i_2)\lambda_1^k\lambda_2^{d-k},$$

$$C(k, i) = \binom{i}{k}.$$

Let $p_u(i_1, i_2)$ denote the probability that in equilibrium $i_1(i_2)$ links in trunk 1(2) are idle. This probability generally depends on u. However, $p_b := p_u(0, 0)$, the blocking probability, is independent of the assignment policy. Calculating this explicitly in terms of the parameters λ_1, λ_2, μ, and for small values of n_1, n_2 suggests the general formula:

Table I—Blocking probabilities

	Policy	p_1	p_2	p_b
$n_1 = 9, n_2 = 9$	u_1	0.178	0.82×10^{-3}	0.502
$y = 10$	u_2	0.84×10^{-3}	0.122	0.502
$n_1 = 9, n_2 = 19$	u_1	0.498	0.807×10^{-3}	0.63×10^{-3}
$y = 20$	u_2	0.14×10^{-3}	0.44×10^{-1}	0.63×10^{-3}

Notation: $y = \lambda\mu^{-1}$, $p_1 = \sum_{i_2} p(0, i_2)$, $p_2 = \sum_{i_1} p(i_1, 0)$

$$p_b^{-1} = \sum_{d=0}^{n_1+n_2} \sum_{i=0}^{d} \gamma(i, d) y_1^{-i} y_2^{-d+i}, \tag{15}$$

where $y_i := \lambda_i \mu^{-1}$, $i = 1, 2$. The coefficients γ are given by the recursion

$$\gamma(i, 0) = 1, \qquad \text{all } i$$

$$\gamma(i, d) = 0, \qquad i > d \quad \text{or} \quad i < 0$$

$$\gamma(i, d+1) = (n_1 - i + 1)\gamma(i-1, d) + (n_2 - d + i)\gamma(i, d).$$

The formula (15) reduces to the Erlang loss formula when $y_1 = y_2$. It is true for all y_1, y_2 and $n_1 + n_2 \leq 3$, and it has been verified for many particular cases. However, a proof of correctness is not available.

Consider now two numerical examples. In both $\epsilon_1 = 0.81$, $\epsilon_2 = 0.7$. Let u_1, respectively u_2, denote the "overflow" policy that assigns every call to trunk 1, respectively trunk 2, if it has an idle link. The table below shows that the probability of blocking a single trunk does vary considerably with policy, but the probability of blocking both trunks does not vary.

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APPENDIX

For $d \geq 1$ define vectors $\mathbf{w}_d = \{\mathbf{w}_d(I), I \subset \Omega\}$ as follows:

$$\mathbf{w}_d(I) = 0, \qquad |I| > d \tag{16}$$

$$\mathbf{w}_d(I) = \prod_{i \in I} \lambda_i, \qquad |I| = d \tag{17}$$

$$\mathbf{w}_d(I) = \lambda_i \mathbf{w}_{d-1}(I) + \lambda_i \mathbf{w}_{d-1}(I - i), \tag{18}$$

where $\quad i := \min\{j \mid j \in I\}, \qquad 0 < |I| < d,$

$$\mathbf{w}_d(\phi) = 0. \tag{19}$$

Lemma 3: $\mathbf{w}_d(I) = \lambda_j \mathbf{w}_{d-1}(I) + \lambda_j \mathbf{w}_{d-1}(I - j),$

$$j \in I, \qquad I \subset \Omega, \qquad d \geq 2. \tag{20}$$

Proof: By direct verification we see that (20) is true for $d = 2$. Suppose it is true for $d \leq e$, and consider $d = e + 1$. If $|I| > e + 1$, (20) is trivial.

Suppose $|I| = e + 1$. Then, from (17),

$$\mathbf{w}_{e+1}(I) = \prod_{i \in I} \lambda_i.$$

On the other hand, if $j \in I$, then

$$\lambda_j \mathbf{w}_e(I) + \lambda_j \mathbf{w}_e(I - j) = 0 + \lambda_j \prod_{i \in I-j} \lambda_i = \prod_{i \in I} \lambda_i$$

and so (20) again holds.

Finally, suppose $0 < |I| < e + 1$, and let $i = \min\{j \mid j \in I\}$. Then by (18),

$$\mathbf{w}_{e+1}(I) = \lambda_i \mathbf{w}_e(I) + \lambda_i \mathbf{w}_e(I - i). \tag{21}$$

If $j = i$ then (20) again holds. Suppose $j \neq i$. By induction hypothesis,

$$\mathbf{w}_e(I) = \lambda_j \mathbf{w}_{e-1}(I) + \lambda_j \mathbf{w}_{e-1}(I - j)$$

$$\mathbf{w}_e(I - i) = \lambda_j \mathbf{w}_{e-1}(I - i) + \lambda_j \mathbf{w}_{e-1}(I - i - j).$$

Substitution in (21) gives

$$\begin{aligned} \mathbf{w}_{e+1}(I) &= \lambda_i \lambda_j [\mathbf{w}_{e-1}(I) + \mathbf{w}_{e-1}(I - j) + \mathbf{w}_{e-1}(I - i) \\ &\quad + \mathbf{w}_{e-1}(I - i - j)] \\ &= \lambda_j [\lambda_i \mathbf{w}_{e-1}(I) + \lambda_i \mathbf{w}_{e-1}(I - i)] + \lambda_j [\lambda_i \mathbf{w}_{e-1}(I - j) \\ &\quad + \lambda_i \mathbf{w}_{e-1}(I - j - i)] \\ &= \lambda_j \mathbf{w}_e(I) + \lambda_j \mathbf{w}_e(I - j), \qquad \text{by induction hypothesis.} \quad \square \end{aligned}$$

Lemma 4: $$\mathbf{w}_d(I + j) = \sum_{r=1}^{d-1} \lambda_j^r \mathbf{w}_{d-r}(I), \qquad j \notin I.$$

Proof: From (20)

$$\mathbf{w}_d(I + j) = \lambda_j \mathbf{w}_{d-1}(I + j) + \lambda_j \mathbf{w}_{d-1}(I).$$

The assertion follows by iterating on the first term on the right. $\square$

Let $N(I, d)$ denote the set of all $|I|$-tuples $n = \{n_i \mid i \in I\}$ such that $n_i \geq 1$ and $\sum n_i = d$.

Lemma 5: $$\mathbf{w}_d(I) = \sum_{n \in N(I,d)} \prod_{i \in I} \lambda_i^{n_i}.$$

Proof: If $d = 1$, the assertion is immediate. Suppose it is true for some $d \geq 1$. From (20) and $j \in I$

$$\mathbf{w}_{d+1}(I) = \lambda_j[\mathbf{w}_d(I) + \mathbf{w}_d(I - j)]$$
$$= \sum_{n \in N(I,d)} \lambda_j \prod_{i \in I} \lambda_i^{n_i} + \sum_{n \in N(I-j,d)} \lambda_j \prod_{i \in I-j} \lambda_i^{n_i}.$$

The first term is the sum over all $n \in N(I, d + 1)$ with $n_j > 1$, while the second term is the sum over all $n \in N(I, d + 1)$ with $n_j = 1$. Hence the assertion is true for $d + 1$. □

Lemma 6: $$(d - |I|)\mathbf{w}_d(I) = \sum_{r=1}^{d-1} \sum_{j \in I} \lambda_j^{d-r} \mathbf{w}_r(I).$$

Proof: By Lemma 5, the right-hand side equals

$$\sum_{r=1}^{d-1} \sum_{j \in I} \sum_{n \in N(I,r)} \lambda_j^{d-r} \prod_{i \in I} \lambda_i^{n_i}. \tag{22}$$

Each term in (22) is of the form

$$\prod_{i \in I} \lambda_i^{m_i} \tag{23}$$

for some $m \in N(I, d)$. Hence the assertion will be proved if it is shown that each of the terms (23) appears exactly $(d - |I|)$ times in (22). Fix m. Then (23) appears in the sum (22)

$$\sum_{n \in N(I,r)} \lambda_j^{d-r} \prod_{i \in I} \lambda_i^{n_i}$$

if and only if $m_j > d - r$, or $r > d - m_j$. Therefore, the term (23) appears in (22) exactly $\sum_{j \in I} (m_j - 1) = d - |I|$ times, as required. □

Next, for $1 \leq d \leq n$ define vectors $\{\mathbf{z}_d(I), I \in \Omega\}$ as follows:

$$\mathbf{z}_1(I) = \sum_{i \in I} \lambda_i \tag{24}$$

$$\mathbf{z}_d(I) = 0, \qquad 0 \leq |I| < d \tag{25}$$

$$\mathbf{z}_d(I + j) = \lambda_j \mathbf{z}_{d-1}(I) + \mathbf{z}_d(I). \tag{26}$$

Lemma 7: Let $d = |J|$ and $I \subset J$. Then for $e \geq 1$

$$\mathbf{w}_{d+e}(I) + \sum_{r=1}^{d} (-1)^r \mathbf{z}_r(J) \mathbf{w}_{d+e-r}(I) = 0.$$

Proof: If $I = \phi$ the assertion is trivial by (19). We first prove the assertion for $I = J \neq \phi$ using induction on d. If $d = 1$ and $I = \{i\}$, then

$$\mathbf{w}_{1+e}(I) = \lambda_i \mathbf{w}_e(I), \qquad \text{by (20)}$$
$$= z_1(I)\mathbf{w}_e(I), \qquad \text{by (24).}$$

Suppose the assertion is true for 1, $\cdots$, d. Let $|I| = d$ and $j \notin I$. Then by (20),

$$\mathbf{w}_{d+1+e-r}(I + j) - \lambda_j \mathbf{w}_{d+e-r}(I + j) = \lambda_j \mathbf{w}_{d+e-r}(I).$$

Multiplying both sides by $\mathbf{z}_r(J)$ for $r \geq 1$ and summing for $r = 0, \cdots, d + 1$ gives

$$\mathbf{w}_{d+1+e}(I + j) - \lambda_j \mathbf{w}_{d+e}(I + j)$$
$$+ \sum_{i=1}^{d+1} (-1)^r \mathbf{z}_r(J)[\mathbf{w}_{d+1+e-r}(I + j) - \lambda_j \mathbf{w}_{d+e-r}(I + j)]$$
$$= \lambda_j[\mathbf{w}_{d+e}(I) + \sum_{r=1}^{d+1} (-1)^r \mathbf{z}_r(J) \mathbf{w}_{d+e-r}(I)]$$
$$\lambda_j[\mathbf{w}_{d+e}(I) + \sum_{r=1}^{d} (-1)^r \mathbf{z}_r(J) \mathbf{w}_{d+e-r}(I)],$$

since $\mathbf{z}_{d+1}(J) = 0$ by (25)

$$= 0 \text{ by induction hypothesis.}$$

Hence,

$$0 = \mathbf{w}_{d+1+e}(I + j) - \lambda_j \mathbf{w}_{d+e}(I + j) + \sum_{r=2}^{d+1} (-1)^r [\mathbf{z}_r(J)$$
$$+ \lambda_j \mathbf{z}_{r-1}(J)] \mathbf{w}_{d+1+e-r}(I + j) - \mathbf{z}_1(J) \mathbf{w}_{d+e}(I + j)$$
$$- (-1)^{d+1} \mathbf{z}_{d+1}(J) \lambda_j \mathbf{w}_{e-1}(I + j)$$
$$= \mathbf{w}_{d+1+e}(I + j) + \sum_{r=1}^{d+1} (-1)^r \mathbf{z}_r(J + j) \mathbf{w}_{d+1+e-r}(I + j),$$

using (26) and the fact that $\mathbf{z}_{d+1}(J) = 0$ by (25).

This completes the proof for the case when $|I| = |J|$. For the case $|I| < |J|$ the proof proceeds as in the last step using induction on $|J|$ for fixed I. □

AUTHOR

Bhaskarpillai Gopinath, M.Sc. (Mathematics), 1964, University of Bombay; Ph.D. (E.E.), 1968, Stanford University; Research Associate, Stanford University, 1967–1968; Alexander von Humboldt Research Fellow, University of Göttingen, 1971–1972; Gordon McKay Professor, University of California, Berkeley, 1980–1981; AT&T Bell Laboratories, 1968–1983. Present affiliation Bell Communications Research, Inc. Mr. Gopinath is engaged in research in communications and computer science.