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COVER SHEET FOR TECHNICAL MEMORANDUM

TITLE- Analysis on the Operation of the Fine Tone Tracking Loop of the LM VHF Ranging Transponder

TM- 68-2034-2

DATE- February 1, 1968

FILING CASE NO(S)- 320

AUTHOR(S)- K. H. Schmid

FILING SUBJECT(S)- Fine Tone Tracking
(ASSIGNED BY AUTHOR(S))- Loop - Ranging Transponder

(NASA-CR-95460) ANALYSIS ON THE OPERATION OF THE FINE TONE TRACKING LOOP OF THE LM VHF RANGING TRANSPONDER (Bellcomm, Inc.) 48 p N79-72637

Unclas
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(NASA CR OR TMX OR AD NUMBER) (CATEGORY)

[redacted]

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This particular analysis pertains to the case where the locally generated, fine tone gating signal is phase modulated by $\pm 1/8$ cycle at a modulating frequency of 1/4 of the fine tone frequency, and where the receiver crystal bandpass filter is sufficiently narrow to pass only the first pair of error signal sidebands which are centered on the IF carrier frequency.

Results show that with suitable filtering, the proposed tracking loop does regenerate a coherent fine tone in either mode. The error signal at the input to the VCXO is such that it adjusts the phase difference between the VCXO output fine tone and the received fine tone so that the tracking phase error tends to zero. It is also shown that the ranging plus voice mode desensitizes the error signal characteristic, i.e., reduces the slope of the error signal at crossover. Thus, the loop becomes more susceptible to noise including voice frequency harmonics, and other perturbations; this effect results in a decrease in range accuracy when using this mode.

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1100 Seventeenth Street, N.W. Washington, D.C. 20036

SUBJECT: Analysis on the Operation of the Fine Tone Tracking Loop of the LM VHF Ranging Transponder - Case 320

DATE: February 1, 1968

FROM: K. H. Schmid

TM-68-2034-2

TECHNICAL MEMORANDUM

I. INTRODUCTION

In the planned mission configuration the Apollo Lunar Module (LM) will be required to rendezvous with the orbiting Command and Service Module (CSM). Optical techniques, and an X-band radar having a range readout aboard the LM, will be used to accomplish the rendezvous. In an emergency, however, the CSM may be required to rendezvous with the LM. A VHF ranging system, providing a range readout aboard the CSM, is being developed to add this capability to the overall system.

The new ranging system consists of a Digital Ranging Generator (DRG) which is to interface with a modified VHF transceiver aboard the CSM, and a Range Tone Transfer Assembly (RTTA) which is to interface with a similar transceiver aboard the LM. The transceivers will be used to establish the two-way communications link between the LM and CSM.

The DRG generates coarse (low frequency), mid (medium frequency), and fine (high frequency) square wave tones which are used to ON-OFF key (100% AM) the CSM transmit carrier. Tones are selected according to a particular sequence most suitable for acquisition and lock-up of the ranging system; the sequential selection of tones is performed automatically in the DRG.

The LM ranging transponder, consisting of the VHF transceiver and the RTTA, as shown in Figure 1, receives and detects the tones. Note that the coarse tone is added (modulo-2) to the mid tone when the coarse tone is tranmitted over the VHF links. If a mid tone or mid and coarse tone is detected at the LM, it is simply re-modulated on the return VHF link to the CSM.

If the fine tone is received at the LM, however, a fine tone tracking loop (see Figure 2) is employed to regenerate the fine tone prior to modulation of the return link to the CSM. By using this narrowband loop to phase lock the local voltage controlled crystal oscillator (VCXO) to the received fine tone signal, a relatively noise-free fine tone, which is coherent with the received fine tone, can be generated for transmission to the CSM.

The DRG compares the phase between its transmitted tones and the tones received from the LM. The DRG derives the range number from this comparison and presents a range readout to the external display and computer.

In addition to the ranging alone mode described above, the new system provides a voice plus ranging mode. Simultaneous voice and ranging is achieved by clipping the voice and using the resultant bipolar waveform to gate the ranging fine tone on and off. Thus the fine tone is inhibited when the transmit clipped voice waveform is negative and is passed when the transmit clipped waveform is positive. It should be noted that voice transmission is not permissible while a mid tone or a mid and coarse tone is being transmitted during the acquisition phase of the ranging sequence.

It is the purpose of this memorandum to verify that the LM fine tone tracking loop, in fact, will track the received fine tone during either the ranging or ranging plus voice mode. This objective is accomplished by deriving the characteristic of the error signal (vs. tracking phase error) at the input to the VCXO. The derivations of the error signal for the ranging and ranging plus voice modes are performed in Appendix A and Appendix B, respectively.

Major results of the derivations are summarized and discussed below.

II. FINE TONE TRACKING LOOP SIGNALS (NO VOICE MODULATION)

A. The Received Signal

Once the ranging system is locked-up, the fine tone is used exclusively to key the CSM transmitter. The fine tone keying waveform , $f_1(t)$, is shown in Figure 3a. Thus the received signal, $f_2(t)$, at the input to the LM receiver gate is the carrier signal ON-OFF modulated at the fine tone frequency (see Figure 3b). For convenience, also refer to Figure 2, which depicts the fine tone tracking loop and the specific locations of signals discussed here.

B. The Gating Signal

The gating signal, $f_3(t)$, is formed locally by using digital circuits to divide the VCXO frequency down to the fine tone frequency. The resultant signal is then "early/late" phase shifted by $\phi_2(t)$. The phase shift, $\phi_2(t)$, as shown in Figure 3c, shifts the locally generated fine tone by $\pm 1/8$ cycle (referenced to the fine tone frequency). The early/late ($\pm 1/8$ cycle) phase shifting is performed at a rate of 1/4 the fine tone frequency, i.e., $w_\phi = {}^{w}m/4$, where w_ϕ is the rate of phase switching and w_m is the fundamental frequency of the fine tone (radians per second). Note that the analysis presented in Appendices A and B is only applicable to the case where $w_\phi = {}^{w}m/4$. Other switching frequencies may be feasible but they are not explored here.

C. The IF Signals

The signal at the output of the gate is given by $f_4(t)$. As a result of the gating process, $f_4(t)$ contains a wide spectrum of frequencies centered about the IF carrier.

The crystal bandpass filter is designed to pass only the carrier (at w_c) and the first pair of sidebands (at $w_c \pm w_\phi$); all other sidebands are rejected. The crystal filter output signal, $f_5(t)$, is fed to the IF amplifier to produce the input signal to the envelope detector, $f_6(t)$.

D. Envelope Detector Output Signal

Demodulation of the IF signal is accomplished at the envelope detector. The output signal, $f_7(t)$, is sent to the synchronous detector, where it is multiplied by a coherent reference signal, $f_{REF}(t)$. It has been shown in Appendix A that $f_7(t)$ consists of one component in-phase ($f_i(t)$) with $f_{REF}(t)$ and a second component in quadrature ($f_q(t)$) with $f_{REF}(t)$. The in-phase component, $f_i(t)$, is of great value since its amplitude is proportional to the tracking phase error, ϕ_1, for small angles of ϕ_1. All the above signals are at a radian frequency of $w_\phi = {}^{w}m/4$.

For purposes of clarity, the equations for these signals are summarized below:

a) Reference Signal Input to Synchronous Detector

$$f_{REF}(t) = \cos \frac{w_m t}{4} = \cos w_\phi t$$

b) Envelope Detector Output Signal (Input to Synchronous Detector)

$$f_7(t) = f_i(t) + f_q(t) + \text{DC term}$$

c) In-phase Component of $f_7(t)$

$$f_i(t) = 1.299K_1 \cos \frac{w_m t}{4} \left(\sin\phi_1 + 0.112 \sin 3\phi_1 - 0.038 \sin 5\phi_1 \ldots \right)$$

d) Quadrature Component of $f_7(t)$

$$f_q(t) = 0.471K_1 \sin \frac{w_m t}{4}$$

$$+ 0.062K_1 \sin \frac{w_m t}{4} \left(\cos\phi_1 + 0.660 \cos 3\phi_1 + 0.134 \cos 5\phi_1 \ldots \right)$$

where K_1 = constant

w_m = fundamental frequency of the fine tone (rad/sec)

w_ϕ = $^{w}m/4$ = early/late switching frequency (rad/sec)

t = time (seconds)

ϕ_1 = $w_m \tau$ = tracking phase error (see Figure 3a)

E. Error Signal at Input to the VCXO

The error signal at the input to the VCXO is derived by multiplying $f_7(t)$ by the reference signal, $f_{REF}(t)$, and then low pass filtering the resultant signal. When this is done, no contribution from the quadrature component of $f_7(t)$ is obtained at the output of the filter; however, a "DC" voltage proportional to ϕ_1 is derived from the in-phase component of $f_7(t)$. Thus the synchronous detector and low pass filter acts as a coherent amplitude detector for $f_i(t)$. As mentioned previously, the amplitude of $f_i(t)$ is proportional to ϕ_1 for small angles of ϕ_1.

The resultant error voltage as a function of ϕ_1 has been derived in Appendix A and is repeated here for convenience.

$$V(\phi_1) = 0.093\ A_1\ (\sin \phi_1 + 0.112 \sin 3\phi_1 - 0.038 \sin 5\phi_1 \ldots)$$

This function is plotted in Figure 4. Note that the error voltage is zero for $\phi_1 = 0$ and changes polarity when ϕ_1 changes polarity. Thus $V(\phi_1)$ is a usable error voltage for phase locking the VCXO to the received fine tone signal.

III. FINE TONE TRACKING LOOP SIGNALS (WITH VOICE MODULATION)

A. The Received Signal

When simultaneous voice and ranging is transmitted by the CSM, an intermittent series of fine tone pulses (see Figure 3d), amplitude modulated on the carrier, is received at the input to the LM receiver gate. Let this new input waveform be designated by $f_{2v}(t)$ where the "v" subscript hereafter denotes the voice mode of operation.

It is now desired to verify that the fine tone tracking loop will remain in lock when the intermittent series of fine tone pulses is received instead of the continuous series of fine tone pulses. The derivation of the error at the input to the VCXO under these conditions has been performed in Appendix B. For the initial analysis the clipped voice waveform is represented by a rectangular waveform with a fundamental frequency of w_v radians per second. The results are generalized at the conclusion of the analysis for a clipped voice waveform having random pulse widths.

B. The Gating Signal

The gating signal is identical to the gating signal described in Section II.

C. The IF Signals

The signal at the output of the gate is given by $f_{4v}(t)$. As a result of the gating process, $f_{4v}(t)$ contains a large number of sidebands centered about the IF carrier.

The crystal bandpass filter passes the carrier (at w_c) and the first pair of significant sidebands (at $w_c \pm w_\phi$); in addition, sideband energy due to the ON-OFF voice modulation is passed by the filter. The crystal filter output signal, $f_{5v}(t)$, is amplified in the IF amplifier to produce the input signal to the envelope detector, $f_{6v}(t)$.

D. Envelope Detector Output Signal

Demodulation of the IF signal is accomplished at the envelope detector. Again, the output signal, $f_{7v}(t)$, contains an in-phase and quadrature component (compared to the phase of the reference signal $f_{REF}(t)$) plus other sideband energy from the clipped voice modulation.

Equations for these signals are summarized below:

a) Reference Signal Input to Synchronous Detector

$$f_{REF}(t) = \cos \frac{w_m t}{4} = \cos w_\phi t$$

b) Envelope Detector Output Signal (Input to Synchronous Detector)

$$f_{7v}(t) = f_{iv}(t) + f_{qv}(t) + f_{rv}(t)$$

c) In-phase Component of $f_{7v}(t)$

$$f_{iv}(t) = 0.650K_1 \cos \frac{w_m t}{4} \left(\sin \phi_1 + 0.112 \sin 3\phi_1 - 0.038 \sin 5\phi_1 \ldots \right)$$

d) Quadrature Component of $f_{7v}(t)$

$$f_{qv}(t) = 0.236K_1 \sin \frac{w_m t}{4} + 0.031\ K_1 \sin \frac{w_m t}{4} \left(\cos \phi_1 + 0.660 \cos 3\phi_1 + 0.134 \cos 5\phi_1 \ldots \right)$$

e) Remaining Components of $f_{7v}(t)$: DC term and terms caused by voice modulation, i.e.

$f_{rv}(t)$ = remaining terms of lesser importance (see Appendix B).

Note that $f_{iv}(t) = 1/2\ f_i(t)$; $f_{qv}(t) = 1/2\ f_q(t)$

E. Error Signal at Input to VCXO

The error signal at the input to the VCXO is derived by multiplying $f_{7v}(t)$ by the reference signal, $f_{REF}(t)$, and then low pass filtering the resultant signal. Again, the error signal is obtained solely from the in-phase component, $f_{iv}(t)$. From the nature of the equations for $f_{rv}(t)$ derived in Appendix B for a rectangular clipped voice waveform, it is evident that the low pass filter will reject all significant amounts of energy due to $f_{rv}(t)$ when a random pulse width, clipped voice waveform is assumed.

Since $f_{iv}(t) = 1/2\ f_i(t)$ as shown in the previous paragraph, the error signal (with voice modulation) is given by:

$$V_v(\phi_1) = 0.047\ A_1\ (\sin\phi_1 + 0.112 \sin 3\phi_1 - 0.038 \sin 5\phi_1 \ldots)$$

$$= 1/2\ V(\phi_1)$$

This function is plotted in Figure 4. Thus the error signal characteristic with voice modulation exhibits 1/2 the slope and amplitude that would exist with no voice modulation of the ranging fine tone.

IV. CONCLUSIONS

Results based on the analysis performed in Appendices A and B, as summarized and discussed in the previous section of this memorandum, show that a usable error signal characteristic at the input to the VCXO is obtained for either the ranging mode or the ranging plus voice mode. The error signal is usable since the error voltage is zero when $\phi_1 = 0$ (i.e., zero error voltage is produced when the VCXO is tracking the received fine tone perfectly) and since the error signal changes polarity when ϕ_1 changes polarity. Thus the VCXO frequency always will be adjusted by the error signal such that $\phi_1 \rightarrow 0$.

It has been shown that the slope of the error signal in the voice plus ranging mode is 1/2 the slope in the ranging mode. Thus the addition of voice modulation causes the LM fine tone tracking loop to be more susceptible to noise and other perturbations. An accompanying decrease in range accuracy is to be expected when the voice plus ranging mode is selected.

K. H. Schmid

2034-KHS-nep K. H. Schmid

Attachments

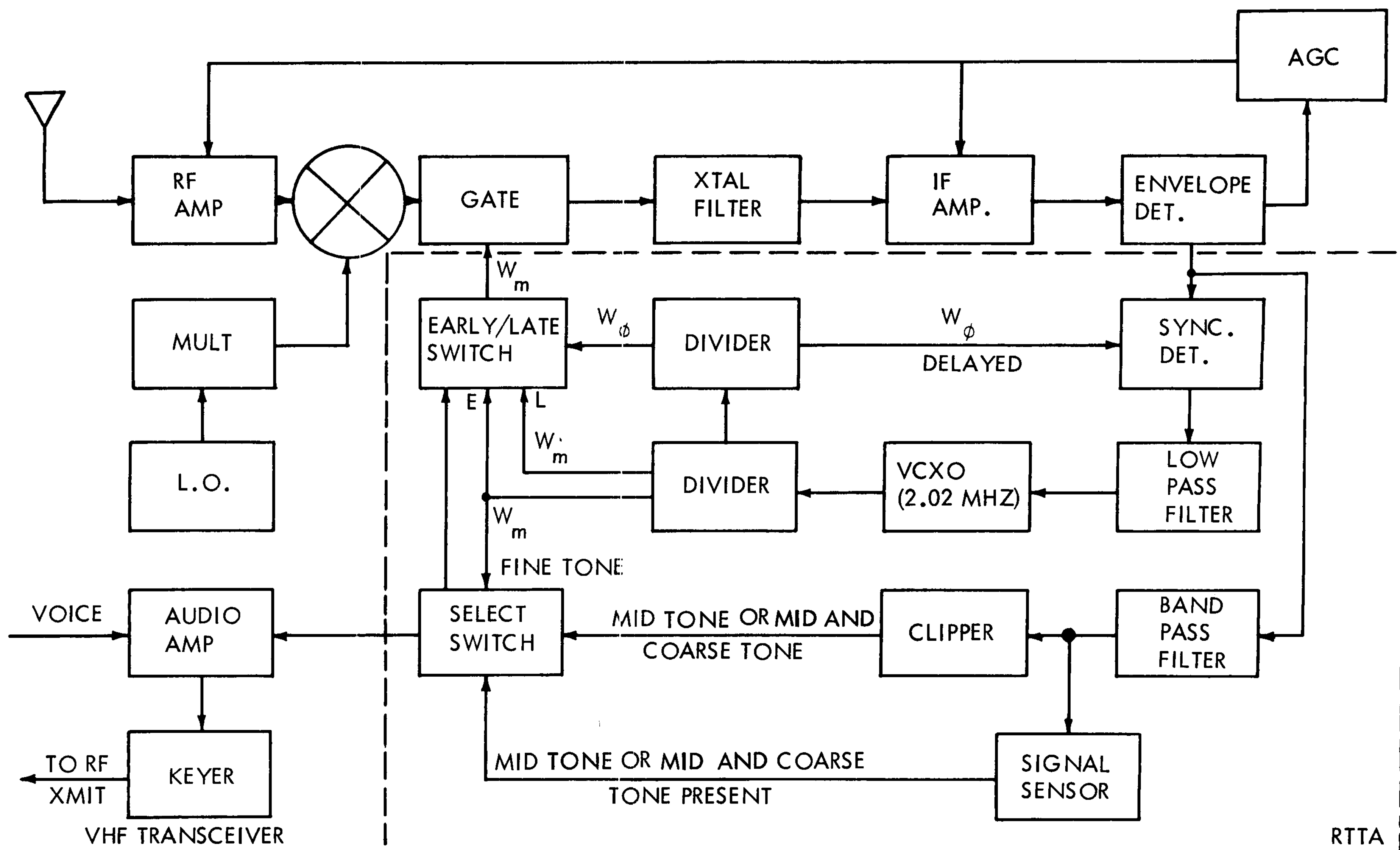
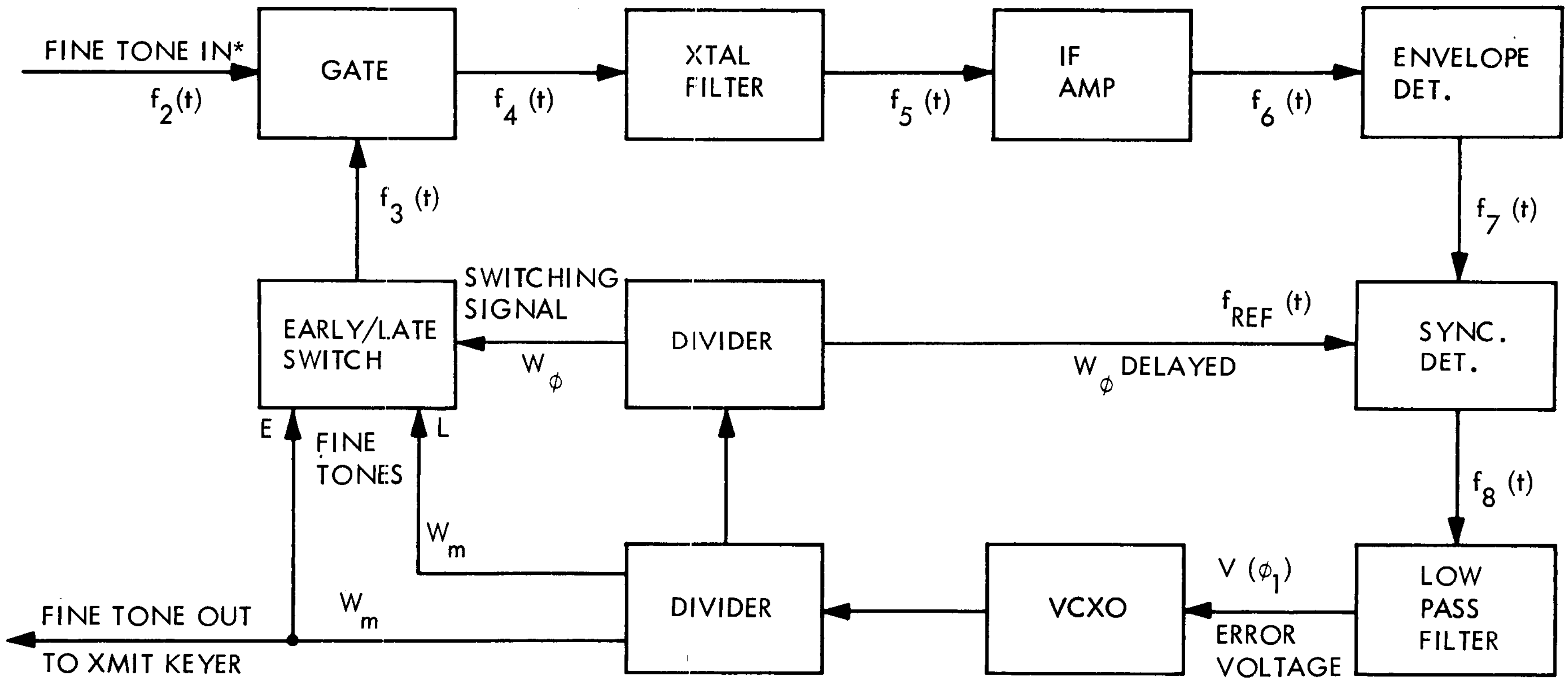
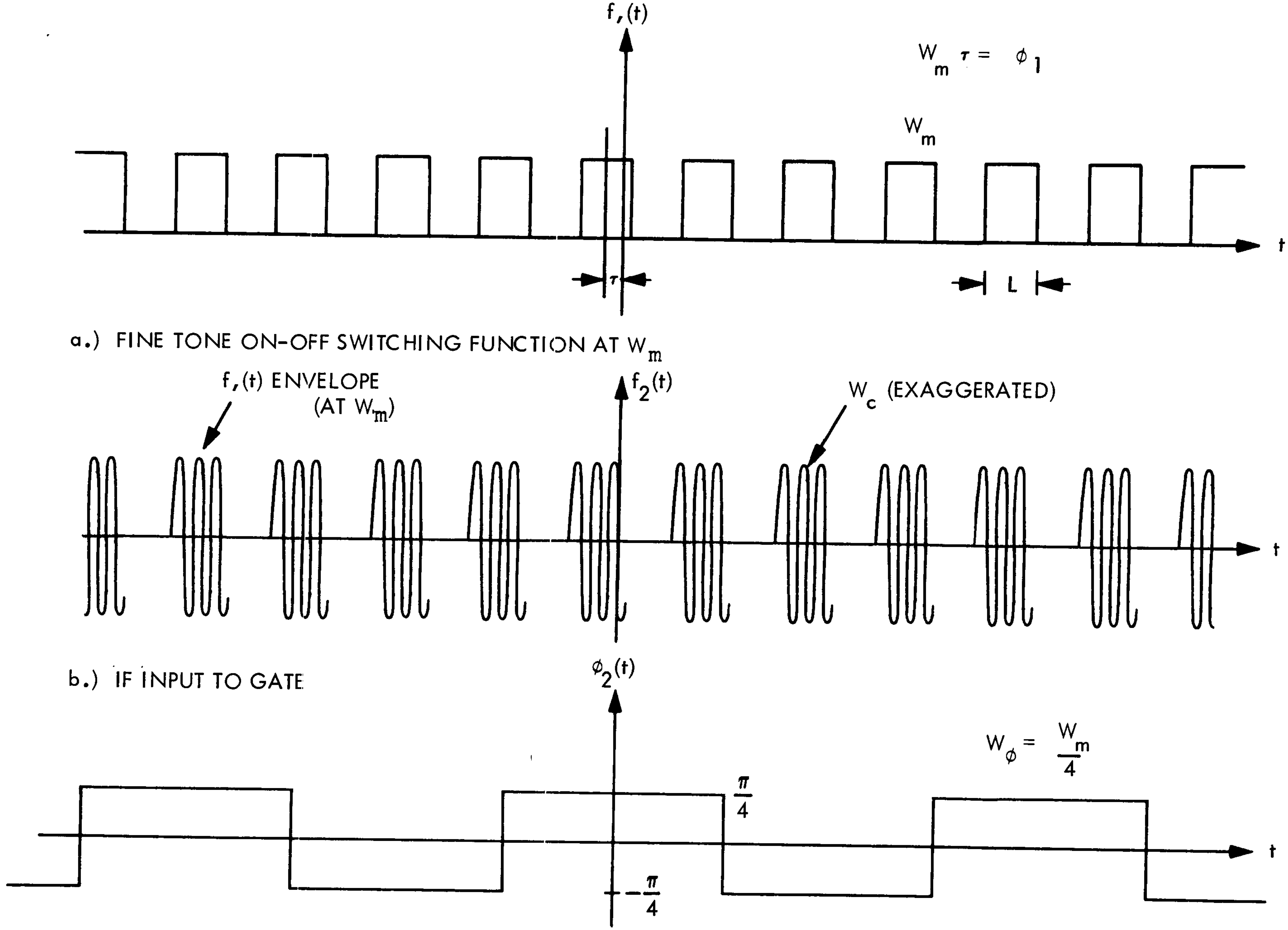


FIGURE I - LM VHF RANGING TRANSPONDER



*FINE TONE IN IS AMPLITUDE MODULATED ON IF.

FIGURE 2. FINE TONE TRACKING LOOP

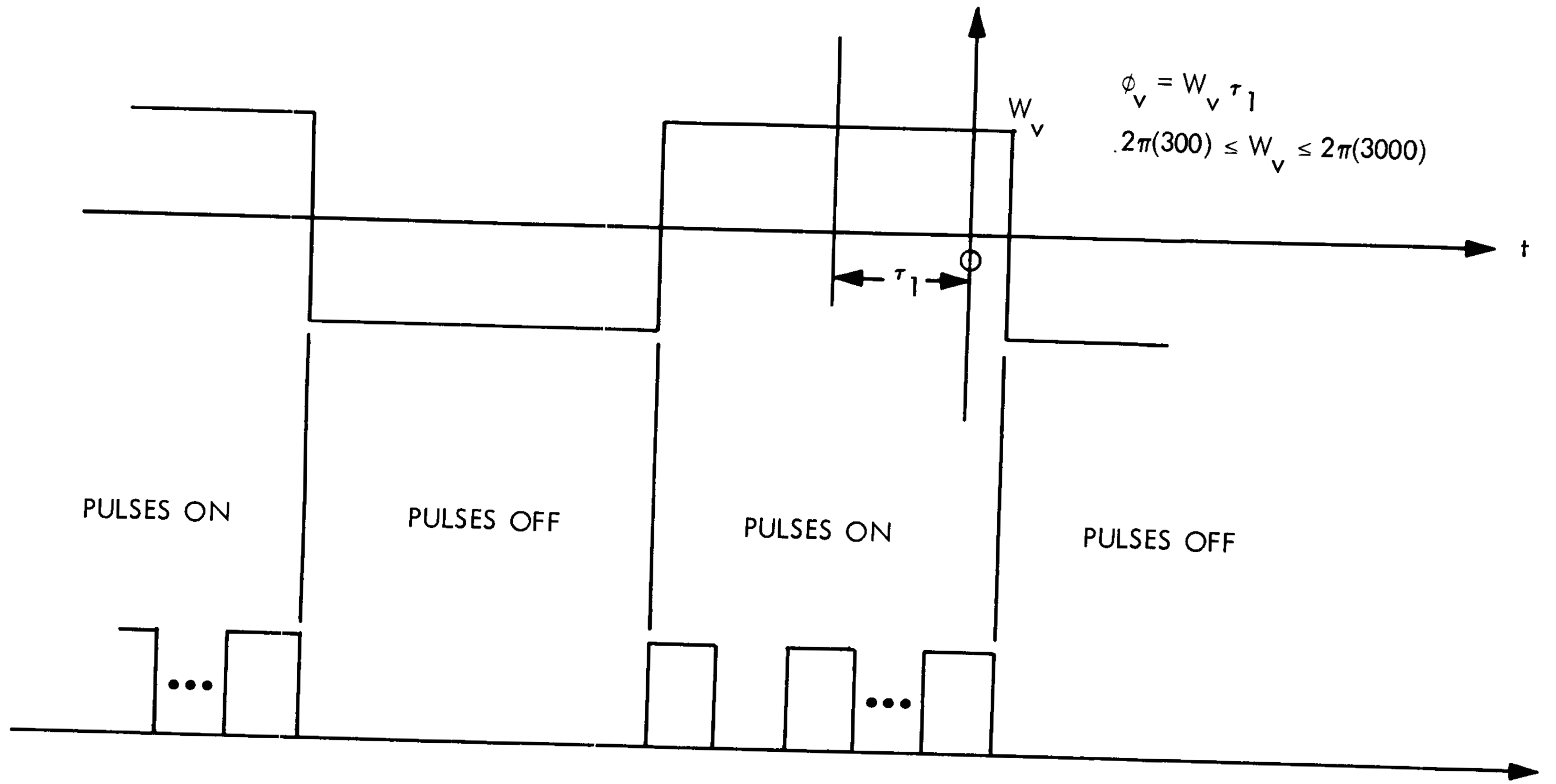


a.) FINE TONE ON-OFF SWITCHING FUNCTION AT W_m

b.) IF INPUT TO GATE

c.) PHASE OF FINE TONE GATING SIGNAL AT OUTPUT OF EARLY/LATE SWITCH

FIGURE 3. FINE TONE TRACKING LOOP WAVEFORMS



d. CLIPPED VOICE WAVEFORM (TOP) AND RESULTANT INTERMITTENT SERIES OF FINE TONE PULSES (BOTTOM).

FIGURE 3. (CONTINUED)

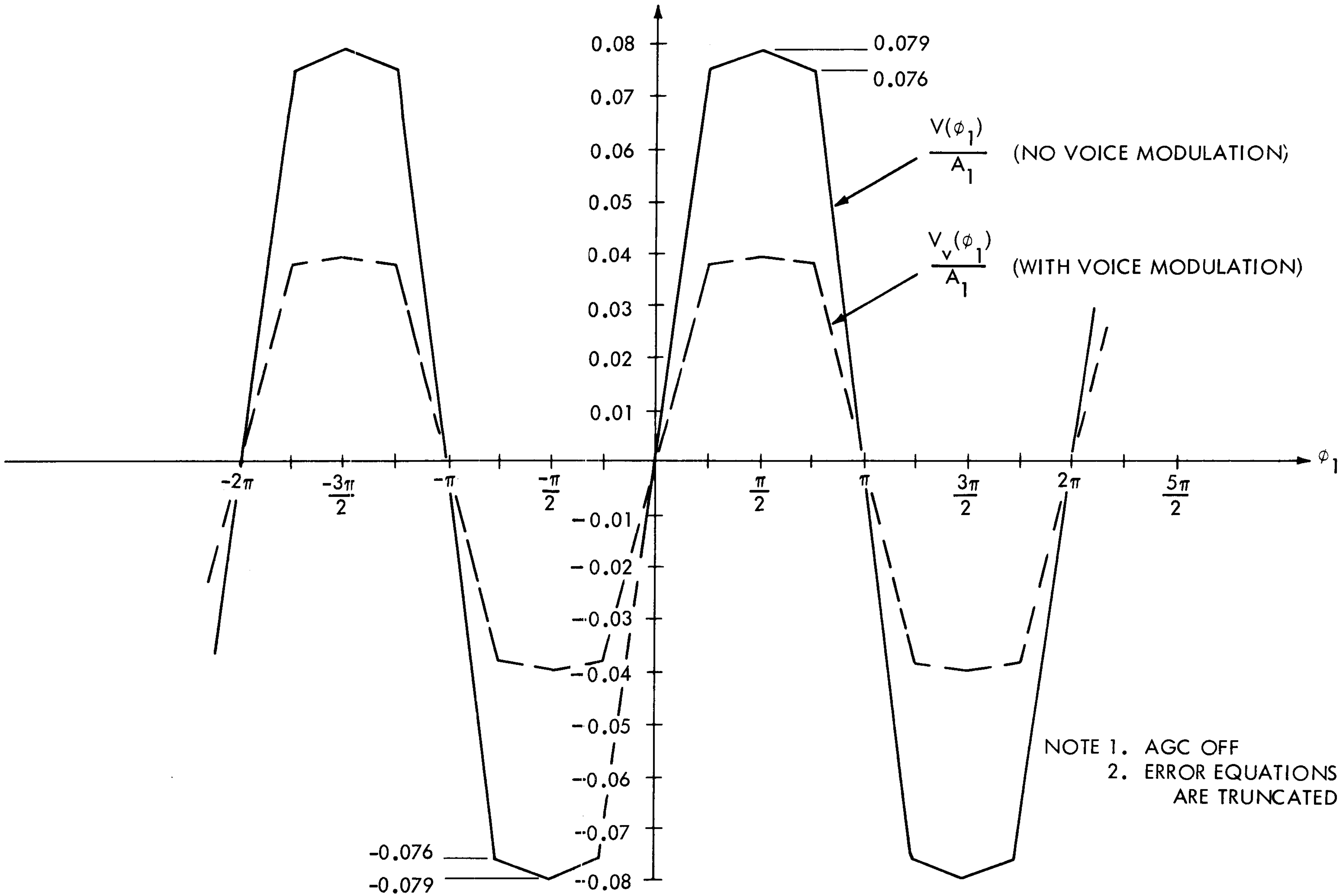


FIGURE 4. ERROR SIGNALS AT INPUT TO VCXO.

APPENDIX A

ERROR SIGNAL FOR LM FINE TONE TRACKING LOOP (NO VOICE MODULATION)

1.0 INTRODUCTION

When no voice modulation is present, the received ranging signal at the input to the LM receiver consists of a continuous series of fine tone pulses amplitude modulated on the carrier. The error signal generated by the LM fine tone tracking loop under these signal conditions is derived in this Appendix.

Equations for the received signal and the gating signal are derived first. Then the gate output signal is derived and scrutinized to determine which of its frequency components pass the IF crystal bandpass filter. Additional signal analysis is performed to determine the effect of the IF amplifier, envelope detector, synchronous detector and low-pass filter which follow the IF crystal filter.

The resultant output error signal is shown to be usable for phase locking the locally generated fine tone (derived from the VCXO) to the received fine tone. Thus the LM fine tone tracking loop performs as desired.

2.0 FOURIER SERIES EXPANSION OF FINE TONE WAVEFORM

The Fourier series expansion of the fine tone ON-OFF switching function is derived below. No voice modulation is present; therefore the fine tone consists of a continuous pulse train at w_m (see Figure 3a of the main text for waveform parameters).

$$f_1(t) = \frac{a_o}{2} + \sum_{m=1}^{\infty} c_m \cos\left(\frac{m\pi t}{L} + \phi_m\right)$$

where:

$$c_m = \sqrt{a_m^2 + b_m^2}$$

Appendix A (contd.)

$$\phi_m = \tan^{-1}\left(\frac{-b_m}{a_m}\right)$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f_1(t) \cos \frac{m\pi t}{L} dt$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f_1(t) \sin \frac{m\pi t}{L} dt$$

Therefore,

$$a_m = \frac{1}{L}\int_{-\tau-\frac{L}{2}}^{-\tau+\frac{L}{2}} 1 \cdot \cos \frac{m\pi t}{L} dt$$

$$a_m = \frac{2}{m\pi} \cos \frac{-m\pi\tau}{L} \sin \frac{m\pi}{2} \quad , \quad \text{But } w_m L = \pi$$

$$a_m = \frac{2}{m\pi} \sin \frac{m\pi}{2} \cos(-m w_m \tau) \quad , \quad \text{Let } w_m \tau = \phi_1$$

$$\therefore a_m = \frac{2}{m\pi} \sin \frac{m\pi}{2} \cos m\phi_1$$

$$b_m = \frac{1}{L}\int_{-\tau-\frac{L}{2}}^{-\tau+\frac{L}{2}} 1 \cdot \sin \frac{m\pi t}{L} dt$$

$$b_m = \frac{2}{m\pi} \sin \frac{m\pi}{2} \sin \frac{-m\pi\tau}{L} \quad , \quad \text{But } w_m L = \pi$$

$$w_m \tau = \phi_1$$

$$\therefore b_m = \frac{-2}{m\pi} \sin \frac{m\pi}{2} \sin m\phi_1$$

Appendix A (contd.)

$$c_m = \frac{2}{m\pi} \sin \frac{m\pi}{2} \sqrt{\cos^2 m\phi_1 + \sin^2 m\phi_1}$$

$$\therefore c_m = \frac{2}{m\pi} \sin \frac{m\pi}{2}$$

$$\phi_m = \tan^{-1}\left(\frac{\sin m\phi_1}{\cos m\phi_1}\right)$$

$$\therefore \phi_m = m\phi_1$$

Thus,

$$f_1(t) = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{2}{m\pi} \sin \frac{m\pi}{2} \cos (mw_m t + m\phi_1)$$

3.0 ON-OFF SWITCHING OF IF CARRIER AT INPUT TO THE GATE

The IF signal is keyed ON-OFF by the fine tone. The equation for this signal is simply the product of the fine tone signal and the IF carrier as follows:

$$f_2(t) = A \cos w_c t \cdot f_1(t)$$

where:

A = amplitude of IF carrier

w_c = IF frequency (radians per sec.)

$$\therefore f_2(t) = \frac{A}{2} \cos w_c t \left[1 + 2 \sum_{m=1}^{\infty} \frac{\sin \frac{m\pi}{2}}{\frac{m\pi}{2}} \cos(mw_m t + m\phi_1)\right]$$

Appendix A (contd.)

4.0 THE GATING SIGNAL

The signal $f_2(t)$ is gated by a phase modulated, locally generated fine tone, $f_3(t)$. The phase modulation is given by $\phi_2(t)$ (see Figure 3c), which is equivalent to $\pm \frac{1}{8}$ cycle early-late phase modulation of the fine tone at a switching rate of w_ϕ radians per second. The resultant gating signal is given by:

$$f_3(t) = \frac{1}{2}\left[1 + 2 \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{\frac{n\pi}{2}} \left(\cos n w_m t + n\phi_2(t)\right)\right]$$

Note that the time reference has been chosen such that t=0 occurs in the center of a $\phi_2(t)$ pulse; all other pertinent waveforms are given an arbitrary time shift from t=0.

5.0 SIGNAL AT THE INPUT OF THE CRYSTAL FILTER

The signal into the crystal filter (output of the gate) is given by $f_4(t)$, (see Figure 2).

$$f_4(t) = f_2(t) \cdot f_3(t)$$

$$\therefore f_4(t) = \frac{A}{4} \cos w_c t \left[1 + 2 \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{\frac{n\pi}{2}} \cos \left(n w_m t + n\phi_2(t)\right) \right.$$

$$+ 2 \sum_{m=1}^{\infty} \frac{\sin \frac{m\pi}{2}}{\frac{m\pi}{2}} \cos(m w_m t + m\phi_1)$$

$$+ 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{\frac{n\pi}{2}} \cdot \frac{\sin \frac{m\pi}{2}}{\frac{m\pi}{2}}$$

$$\left. \cdot \cos(n w_m t + n\phi_2) \cos(m w_m t + m\phi_1)\right]$$

6.0 SIGNAL AT THE OUTPUT OF THE CRYSTAL FILTER

6.1 General

The crystal filter is a band-pass filter centered at w_c. It is designed to pass frequencies at $w_c \pm w_\phi = w_c \pm \frac{w_m}{4}$; it will not pass frequencies at $w_c \pm w_m$.

Each of the terms of $f_4(t)$ are now examined to determine which components pass the filter.

6.2 First Term of $f_4(t)$

$$g_o(t) = \frac{A}{4} \cos w_c t$$

This term represents an IF carrier component which passes the filter.

6.3 Sidebands from Second Term of $f_4(t)$

$$g_1(t) = \frac{A}{2} \cos w_c t \sum_{n=1}^{\infty} \rho_n \cos(nw_m t + n\phi_2)$$

where:

$$\rho_n = \frac{\sin \frac{n\pi}{2}}{\frac{n\pi}{2}} \quad , \quad \rho_n = 0 \text{ for n even} \quad .$$

$$g_1(t) = \frac{A}{2} \cos w_c t \sum_n \rho_n (\cos nw_m t \cos n\phi_2 - \sin nw_m t \sin n\phi_2)$$

Note that $\cos n\phi_2(t) = \gamma_n$ = constant (see Figure 3c). Let $\sin n\phi_2(t) = \alpha_n C(t)$ where $C(t)$ is $\pm$ 1 at a frequency of w_ϕ.

$$\therefore C(t) = 2 \sum_{q=1}^{\infty} \frac{\sin \frac{q\pi}{2}}{\frac{q\pi}{2}} \cos qw_\phi t$$

Appendix A (contd.)

Note that the C(t) pulse is centered on t = 0 since $\phi_2(t)$ is centered on t = 0. Thus,

$$g_1(t) = \frac{A}{2} \cos w_c t \sum_n \left(\rho_n \gamma_n \cos nw_m t - \rho_n \alpha_n C(t) \sin nw_m t \right)$$

Only the last term of $g_1(t)$ can give frequencies that pass the filter; let it equal $g_2(t)$.

$$g_2(t) = -\frac{A}{2} \cos w_c t \sum_n \rho_n \alpha_n \left(2 \sum_q \frac{\sin \frac{q\pi}{2}}{\frac{q\pi}{2}} \cos q w_\phi t \right) \sin nw_m t$$

$$= -A \cos w_c t \sum_n \rho_n \alpha_n \left(\sum_q \delta_q \cos q \frac{w_m t}{4} \right) \sin nw_m t$$

where:

$$\delta_q = \frac{\sin \frac{q\pi}{2}}{q\pi/2} \quad , \quad w_\phi = \frac{w_m}{4} \quad .$$

Only those terms of $g_2(t)$ which produce sidebands at $w_c \pm \frac{w_m}{4}$ are of interest since these frequencies pass the filter. Thus the terms of $g_2(t)$ of interest are those with:

$$\left.\begin{array}{l} q = 4n-1 \\ q = 4n+1 \end{array}\right\} \text{ over all } n \quad .$$

Appendix A (contd.)

Case 1: $q = 4n-1$ over all n

Let these particular terms of $g_2(t)$ be given by $g_3(t)$ and evaluate it for the first few values of n.

$$g_3(t) = -A \cos w_c t \sum_n \rho_n \, \alpha_n \, \delta_{4n-1} \cos\left[\left(4n-1\right) \frac{w_m t}{4}\right] \sin n w_m t$$

TABLE A-1

EVALUATION OF SERIES TERMS OF $g_3(t)$

n	ρ_n	α_n	δ_{4n-1}	$\cos\left(\frac{4n-1}{4} w_m t\right) \sin n w_m t$
1	$2/\pi$	$1/\sqrt{2}$	$-2/3\pi$	$\frac{1}{2} \sin \frac{7 w_m t}{4} - \frac{1}{2} \sin \frac{-w_m t}{4}$
3	$-2/3\pi$	$1/\sqrt{2}$	$-2/11\pi$	$\frac{1}{2} \sin \frac{23 w_m t}{4} - \frac{1}{2} \sin \frac{-w_m t}{4}$
5	$2/5\pi$	$-1/\sqrt{2}$	$-2/19\pi$	$\frac{1}{2} \sin \frac{39 w_m t}{4} - \frac{1}{2} \sin \frac{-w_m t}{4}$
7	$-2/7\pi$	$-1/\sqrt{2}$	$-2/27\pi$	$\frac{1}{2} \sin \frac{55 w_m t}{4} - \frac{1}{2} \sin \frac{-w_m t}{4}$

Thus the signal that passes the filter is given by:

$$g_4(t) = -A \cos w_c t \left(\frac{-1}{2} \sin \frac{-w_m t}{4}\right) \left(\frac{-2\sqrt{2}}{3\pi^2} + \frac{2\sqrt{2}}{33\pi^2} + \frac{2\sqrt{2}}{95\pi^2} - \frac{2\sqrt{2}}{189\pi^2} \cdots\right)$$

$$g_4(t) = \frac{\sqrt{2}A}{\pi^2} \cos w_c t \sin \frac{w_m t}{4} \left(\frac{1}{3} - \frac{1}{33} - \frac{1}{95} + \frac{1}{189} \cdots\right)$$

Appendix A (contd.)

Case 2: $q = 4n+1$ over all n

Let these particular terms of $g_2(t)$ be given by $g_5(t)$ and evaluate it for the first few values of n. Hereafter only the difference frequency component, (which passes the filter) is given in the tables.

$$g_5(t) = -A \cos w_c t \sum_n \rho_n \ \alpha_n \ \delta_{4n+1} \cos\left[\left(4n+1\right) \frac{w_m t}{4}\right] \sin nw_m t$$

TABLE A-2

EVALUATION OF SERIES TERMS OF $g_5(t)$

n	ρ_n	α_n	δ_{4n+1}	Difference Component of $\cos\left(\frac{4n+1}{4}\right)w_m t \sin nw_m t$
1	$2/\pi$	$1/\sqrt{2}$	$2/5\pi$	$-\frac{1}{2}\sin\frac{w_m t}{4}$
3	$-2/3\pi$	$1/\sqrt{2}$	$2/13\pi$	
5	$2/5\pi$	$-1/\sqrt{2}$	$2/21\pi$	
7	$-2/7\pi$	$-1/\sqrt{2}$	$2/29\pi$	
9	$2/9\pi$	$1/\sqrt{2}$	$2/37\pi$	

Thus the signal that passes the filter is given by:

$$g_6(t) = \frac{\sqrt{2}A}{\pi^2} \cos w_c t \sin \frac{w_m t}{4}\left(\frac{1}{5} - \frac{1}{39} - \frac{1}{105} + \frac{1}{203} + \frac{1}{333} \cdots\right)$$

Appendix A (contd.)

6.4 Sidebands From Third Term of $f_4(t)$

$$g_7(t) = \frac{A}{2} \cos w_c t \sum_{m=1}^{\infty} \beta_m \cos(mw_m t + m\phi_1)$$

where:

$$\beta_m = \frac{\sin \frac{m\pi}{2}}{\frac{m\pi}{2}}$$

This signal has sidebands at $w_c \pm mw_m$ and thus none of these sidebands can pass the filter.

6.5 Sidebands From Fourth Term of $f_4(t)$

Examine the fourth term of $f_4(t)$ and set it equal to $h_1(t)$.

$$h_1(t) = A \cos w_c t \sum_n \sum_m \rho_n \beta_m \cos(nw_m t + n\phi_2)\cos(mw_m t + m\phi_1)$$

$$h_1(t) = A \cos w_c t \sum_n \sum_m \rho_n \beta_m \left[\frac{1}{2} \cos(nw_m t + mw_m t + n\phi_2 + m\phi_1) + \frac{1}{2} \cos(nw_m t - mw_m t + n\phi_2 - m\phi_1)\right]$$

$$h_1(t) = \frac{A}{2} \cos w_c t + \sum_n \sum_m \rho_n \beta_m \Big[\cos(nw_m t + mw_m t)\cos(n\phi_2 + m\phi_1) - \sin(nw_m t + mw_m t)\sin(n\phi_2 + m\phi_1) + \cos(nw_m t - mw_m t)\cos(n\phi_2 - m\phi_1) - \sin(nw_m t - mw_m t)\sin(n\phi_2 - m\phi_1)\Big]$$

Note that $\cos(n\phi_2 - m\phi_1)$ is $\neq$ constant unless ϕ_1 is an integral multiple of 2π.

Appendix A (contd.)

$$h_1(t) = \frac{A}{2} \cos w_c t \sum_n \sum_m \rho_n \beta_m \Big\{ \cos(n+m)w_m t$$

$$\cdot \Big[\cos n\phi_2 \cos m\phi_1 - \sin n\phi_2 \sin m\phi_1 \Big]$$

$$- \sin(n+m)w_m t \Big[\sin n\phi_2 \cos m\phi_1 + \cos n\phi_2 \sin m\phi_1 \Big]$$

$$+ \cos(n-m)w_m t \Big[\cos n\phi_2 \cos m\phi_1 + \sin n\phi_2 \sin m\phi_1 \Big]$$

$$- \sin(n-m)w_m t \Big[\sin n\phi_2 \cos m\phi_1 - \cos n\phi_2 \sin m\phi_1 \Big] \Big\}$$

But $\cos n\phi_2(t) = \gamma_n = \text{constant}$; $\sin n\phi_2(t) = \alpha_n\, C(t)$

$$h_1(t) = \frac{A}{2} \cos w_c t \sum_n \sum_m \rho_n \beta_m$$

$$\cdot \Big\{ \cos\Big[(n+m)w_m t\Big] \cos(m\phi_1)\gamma_n - \sin\Big[(n+m)w_m t\Big] \sin(m\phi_1)\gamma_n$$

$$+ \cos\Big[(n-m)w_m t\Big] \cos(m\phi_1)\gamma_n + \sin\Big[(n-m)w_m t\Big] \sin(m\phi_1)\gamma_n \Big\}$$

$$+ \frac{A}{2} \cos w_c t \sum_n \sum_m \rho_n \beta_m$$

$$\cdot \Big\{ - \cos\Big[(n+m)w_m t\Big] \sin(m\phi_1)\alpha_n\, C(t)$$

$$- \sin\Big[(n+m)w_m t\Big] \cos(m\phi_1)\alpha_n\, C(t) + \cos\Big[(n-m)w_m t\Big] \sin(m\phi_1)\alpha_n C(t)$$

$$- \sin\Big[(n-m)w_m t\Big] \cos(m\phi_1)\alpha_n\, C(t) \Big\}$$

Appendix A (contd.)

Only the second term of $h_1(t)$ can produce sidebands that pass the filter. Thus,

$$h_2(t) = A \cos w_c t \sum_n \sum_m \rho_n \beta_m \alpha_n \left\{ \sum_q \delta_q \cos q \frac{w_m t}{4} \right\}$$
$$\cdot \left\{ - \cos \left[(n+m) w_m t \right] \sin m\phi_1 - \sin \left[(n+m) w_m t \right] \cos m\phi_1 \right.$$
$$\left. + \cos \left[(n-m) w_m t \right] \sin m\phi_1 - \sin \left[(n-m) w_m t \right] \cos m\phi_1 \right\}$$

Only those terms of $h_2(t)$ which produce sidebands at $w_c \pm \frac{w_m}{4}$ are of interest since these sidebands pass the filter. Thus the terms of $h_2(t)$ of interest are those with,

$$q = 4(n+m) - 1 \qquad \text{over all } n,\ m$$
$$q = 4(n+m) + 1 \qquad \text{over all } n,\ m$$

$$\left. \begin{aligned} q &= 4(n-m) - 1 \\ q &= 4(n-m) + 1 \end{aligned} \right\} \quad n > m$$

$$\left. \begin{aligned} q &= 4(m-n) - 1 \\ q &= 4(m-n) + 1 \end{aligned} \right\} \quad m > n$$

$$\left. q = 1 \right\} \quad m = n$$

<u>Case 1: $q = 4(n+m) - 1$; all values of m and n</u>

Let these particular terms of $h_2(t)$ be given $h_3'(t)$ and evaluate for the first few values of m and n.

$$h_3'(t) = -A \cos w_c t \sum_n \sum_m \rho_n \beta_m \alpha_n \delta_{4(n+m)-1} \cos \frac{4(n+m)-1}{4} w_m t$$
$$\cdot \left\{ \cos \left[(n+m) w_m t \right] \sin m\phi_1 + \sin \left[(n+m) w_m t \right] \cos m\phi_1 \right\}$$

Note that for these values of q, the other two terms of $h_2(t)$ do not produce frequencies at $w_m/4$.

Appendix A (contd.)

TABLE A-3

EVALUATION OF SERIES TERMS OF $h_3'(t)$

m	n	ρ_n	α_n	β_m	$\delta_{4(n+m)-1}$	Difference Component Of $\cos\frac{4(n+m)-1}{4}w_m t \cos(n+m)w_m t$: $\sin m\phi_1$		Difference Component Of $\cos\frac{4(n+m)-1}{4}w_m t \sin(n+m)w_m t$: $\cos m\phi_1$	
1	1	$2/\pi$	$1/\sqrt{2}$	$2/\pi$	$-2/7\pi$	$\sin\phi_1$		$\cos\phi_1$	
1	3	$-2/3\pi$	$1/\sqrt{2}$	$2/\pi$	$-2/15\pi$	$\sin\phi_1$		$\cos\phi_1$	
1	5	$2/5\pi$	$-1/\sqrt{2}$	$2/\pi$	$-2/23\pi$	$\sin\phi_1$		$\cos\phi_1$	
3	1	$2/\pi$	$1/\sqrt{2}$	$-2/3\pi$	$-2/15\pi$	$\sin 3\phi_1$		$\cos 3\phi_1$	
3	3	$-2/3\pi$	$1/\sqrt{2}$	$-2/3\pi$	$-2/23\pi$	$\sin 3\phi_1$		$\cos 3\phi_1$	
3	5	$2/5\pi$	$-1/\sqrt{2}$	$-2/3\pi$	$-2/31\pi$	$\sin 3\phi_1$	$\frac{1}{2}\cos\frac{-w_m t}{4}$	$\cos 3\phi_1$	$-\frac{1}{2}\sin\frac{-w_m t}{4}$
5	1	$2/\pi$	$1/\sqrt{2}$	$2/5\pi$	$-2/23\pi$	$\sin 5\phi_1$		$\cos 5\phi_1$	
5	3	$-2/3\pi$	$1/\sqrt{2}$	$2/5\pi$	$-2/31\pi$	$\sin 5\phi_1$		$\cos 5\phi_1$	
5	5	$2/5\pi$	$-1/\sqrt{2}$	$2/5\pi$	$-2/39\pi$	$\sin 5\phi_1$		$\cos 5\phi_1$	

Let the difference component contribution from $h_3'(t)$ be represented by $h_3(t)$.

$$h_3(t) = \frac{\sqrt{2}A}{\pi^2} \cos w_c t \cos \frac{w_m t}{4} \left[\sin\phi_1 \left(\frac{2}{7\pi} - \frac{2}{45\pi} - \frac{2}{115\pi} \cdots \right) \right.$$

$$+ \sin 3\phi_1 \left(- \frac{2}{45\pi} + \frac{2}{207\pi} + \frac{2}{465\pi} \cdots \right)$$

$$\left. + \sin 5\phi_1 \left(\frac{2}{115\pi} - \frac{2}{465\pi} - \frac{2}{975\pi} \cdots \right) \cdots \right]$$

$$+ \frac{\sqrt{2}A}{\pi^2} \cos w_c t \sin \frac{w_m t}{4} \left[\cos\phi_1 \left(\frac{2}{7\pi} - \frac{2}{45\pi} - \frac{2}{115\pi} \cdots \right) \right.$$

$$+ \cos 3\phi_1 \left(- \frac{2}{45\pi} + \frac{2}{207\pi} + \frac{2}{465\pi} \cdots \right)$$

$$\left. + \cos 5\phi_1 \left(\frac{2}{115\pi} - \frac{2}{465\pi} - \frac{2}{975\pi} \cdots \right) \cdots \right]$$

Note that if $\phi_1 = k(2\pi)$, $k = 0, \pm 1 \cdots$, $h_3(t)$ reduces to similar form as $g_4(t)$ and $g_6(t)$.

<u>Case 2: $q = 4(n+m)+1$; all values of m and n</u>

Let these particular terms of $h_2(t)$ be given by $h_4'(t)$ and evaluate for the first few values of m and n.

$$h_4'(t) = -A \cos w_c t \sum_n \sum_m \rho_n \, \beta_m \, \alpha_n \, \delta_{4(n+m)+1} \cos \frac{4(n+m)+1}{4} w_m t$$

$$\cdot \left\{ \cos\left[(n+m) w_m t\right] \sin m\phi_1 + \sin\left[(n+m) w_m t\right] \cos m\phi_1 \right\}$$

Appendix A (contd.)

TABLE A-4

EVALUATION OF SERIES TERMS OF $h_4'(t)$

m	n	$\rho_n \cdot \alpha_n \cdot \beta_m$	$\delta_{4(n+m)+1}$	Difference Component Of $\cos\frac{4(n+m)+1}{4}w_m t \cos(n+m)w_m t$: $\sin m\phi_1$		Difference Component Of $\cos\frac{4(n+m)+1}{4}w_m t \sin(n+m)w_m t$: $\cos m\phi_1$	
1	1	$2\sqrt{2}/\pi^2$	$2/9\pi$	$\sin\phi_1$		$\cos\phi_1$	
1	3	$-2\sqrt{2}/3\pi^2$	$2/17\pi$	$\sin\phi_1$		$\cos\phi_1$	
1	5	$-2\sqrt{2}/5\pi^2$	$2/25\pi$	$\sin\phi_1$		$\cos\phi_1$	
3	1	$-2\sqrt{2}/3\pi^2$	$2/17\pi$	$\sin 3\phi_1$	$\frac{1}{2}\cos\frac{w_m t}{4}$	$\cos 3\phi_1$	$-\frac{1}{2}\sin\frac{w_m t}{4}$
3	3	$2\sqrt{2}/9\pi^2$	$2/25\pi$	$\sin 3\phi_1$		$\cos 3\phi_1$	
3	5	$2\sqrt{2}/15\pi^2$	$2/33\pi$	$\sin 3\phi_1$		$\cos 3\phi_1$	
5	1	$2\sqrt{2}/5\pi^2$	$2/25\pi$	$\sin 5\phi_1$		$\cos 5\phi_1$	
5	3	$-2\sqrt{2}/15\pi^2$	$2/33\pi$	$\sin 5\phi_1$		$\cos 5\phi_1$	
5	5	$-2\sqrt{2}/25\pi^2$	$2/41\pi$	$\sin 5\phi_1$		$\cos 5\phi_1$	

Appendix A (contd.)

$$h_4(t) = \frac{\sqrt{2}A}{\pi^2} \cos w_c t \cos \frac{w_m t}{4} \left[\sin\phi_1 \left(-\frac{2}{9\pi} + \frac{2}{51\pi} + \frac{2}{125\pi} \cdots \right) \right.$$

$$+ \sin 3\phi_1 \left(\frac{2}{51\pi} - \frac{2}{225\pi} - \frac{2}{495\pi} \cdots \right)$$

$$\left. + \sin 5\phi_1 \left(-\frac{2}{125\pi} + \frac{2}{495\pi} + \frac{2}{1025\pi} \cdots \right) \cdots \right]$$

$$+ \frac{\sqrt{2}A}{\pi^2} \cos w_c t \sin \frac{w_m t}{4} \left[\cos\phi_1 \left(\frac{2}{9\pi} - \frac{2}{51\pi} - \frac{2}{125\pi} \cdots \right) \right.$$

$$+ \cos 3\phi_1 \left(-\frac{2}{51\pi} + \frac{2}{225\pi} + \frac{2}{495\pi} \cdots \right)$$

$$\left. + \cos 5\phi_1 \left(\frac{2}{125\pi} - \frac{2}{495\pi} - \frac{2}{1025\pi} \cdots \right) \cdots \right]$$

Case 3: $q = 4(n-m)-1$; $n>m$

Let these particular terms of $h_2(t)$ be given by $h_5'(t)$ and evaluate for the first few values of m and n.

$$h_5'(t) = A \cos w_c t \sum_n \sum_m \rho_n \alpha_n \beta_m \delta_{4(n-m)-1} \cos \frac{4(n-m)-1}{4} w_m t$$

$$\cdot \left[\cos(n-m) w_m t \sin m\phi_1 - \sin(n-m) w_m t \cos m\phi_1 \right]$$

Note that for these values of q, the other two terms of $h_2(t)$ do not produce frequencies at $w_m/4$.

Appendix A (contd.)

TABLE A-5

EVALUATION OF SERIES TERMS OF $h_5'(t)$

m	n	ρ_n	α_n	β_m	$\delta_{4(n-m)-1}$	Difference Component Of $\cos\frac{4(n-m)-1}{4}w_m t \cos(n-m)w_m t$: $\sin m\phi_1$		Difference Component Of $\cos\frac{4(n-m)-1}{4}w_m t \sin(n-m)w_m t$: $\cos m\phi_1$	
1	3	$-2/3\pi$	$1/\sqrt{2}$	$2/\pi$	$-2/7\pi$	$\sin\phi_1$		$\cos\phi_1$	
1	5	$2/5\pi$	$-1/\sqrt{2}$	$2/\pi$	$-2/15\pi$	$\sin\phi_1$		$\cos\phi_1$	
1	7	$-2/7\pi$	$-1/\sqrt{2}$	$2/\pi$	$-2/23\pi$	$\sin\phi_1$		$\cos\phi_1$	
3	5	$2/5\pi$	$-1/\sqrt{2}$	$-2/3\pi$	$-2/7\pi$	$\sin 3\phi_1$	$\frac{1}{2}\cos\frac{-w_m t}{4}$	$\cos 3\phi_1$	$-\frac{1}{2}\sin\frac{-w_m t}{4}$
3	7	$-2/7\pi$	$-1/\sqrt{2}$	$-2/3\pi$	$-2/15\pi$	$\sin 3\phi_1$		$\cos 3\phi_1$	
3	9	$2/9\pi$	$1/\sqrt{2}$	$-2/3\pi$	$-2/23\pi$	$\sin 3\phi_1$		$\cos 3\phi_1$	
5	7	$-2/7\pi$	$-1/\sqrt{2}$	$2/5\pi$	$-2/7\pi$	$\sin 5\phi_1$		$\cos 5\phi_1$	
5	9	$2/9\pi$	$1/\sqrt{2}$	$2/5\pi$	$-2/15\pi$	$\sin 5\phi_1$		$\cos 5\phi_1$	

Appendix A (contd.)

$$h_5(t) = \frac{\sqrt{2}A}{\pi^2} \cos w_c t \cos \frac{w_m t}{4} \left[\sin\phi_1 \left(\frac{2}{21\pi} + \frac{2}{75\pi} - \frac{2}{161\pi} \cdots \right) \right.$$

$$+ \sin 3\phi_1 \left(- \frac{2}{105\pi} + \frac{2}{315\pi} + \frac{2}{612\pi} \cdots \right)$$

$$\left. + \sin 5\phi_1 \left(- \frac{2}{245\pi} - \frac{2}{675\pi} \cdots \right) \cdots \right]$$

$$+ \frac{\sqrt{2}A}{\pi^2} \cos w_c t \sin \frac{w_m t}{4} \left[\cos\phi_1 \left(- \frac{2}{21\pi} - \frac{2}{75\pi} + \frac{2}{161\pi} \cdots \right) \right.$$

$$+ \cos 3\phi_1 \left(\frac{2}{105\pi} - \frac{2}{315\pi} - \frac{2}{612\pi} \cdots \right)$$

$$\left. + \cos 5\phi_1 \left(\frac{2}{245\pi} + \frac{2}{675\pi} \cdots \right) \cdots \right]$$

Case 4: $q = 4(n-m)+1;\ n>m$

Let these particular terms of $h_2(t)$ be given by $h_6'(t)$ and evaluate for the first few values of m and n.

$$h_6'(t) = A \cos w_c t \sum_n \sum_m \rho_n\ \alpha_n\ \beta_m\ \delta_{4(n-m)+1} \cos \frac{4(n-m)+1}{4} w_m t$$

$$\cdot \left[\cos(n-m) w_m t \sin m\phi_1 - \sin(n-m) w_m t \cos m\phi_1 \right]$$

Appendix A (contd.)

TABLE A-6

EVALUATION OF SERIES TERMS OF $h_6'(t)$

m	n	ρ_n	α_n	β_m	$\delta_{4(n-m)+1}$	Difference Component Of $\cos\frac{4(n-m)+1}{4}w_m t \cos(n-m)w_m t$: $\sin m\phi_1$		Difference Component Of $\cos\frac{4(n-m)+1}{4}w_m t \sin(n-m)w_m t$: $\cos m\phi_1$	
1	3	$-2/3\pi$	$1/\sqrt{2}$	$2/\pi$	$2/9\pi$	$\sin\phi_1$		$\cos\phi_1$	
1	5	$2/5\pi$	$-1/\sqrt{2}$	$2/\pi$	$2/17\pi$	$\sin\phi_1$		$\cos\phi_1$	
1	7	$-2/7\pi$	$-1/\sqrt{2}$	$2/\pi$	$2/25\pi$	$\sin\phi_1$		$\cos\phi_1$	
3	5	$2/5\pi$	$-1/\sqrt{2}$	$-2/3\pi$	$2/9\pi$	$\sin 3\phi_1$	$\frac{1}{2}\cos\frac{w_m t}{4}$	$\cos 3\phi_1$	$-\frac{1}{2}\sin\frac{w_m t}{4}$
3	7	$-2/7\pi$	$-1/\sqrt{2}$	$-2/3\pi$	$2/17\pi$	$\sin 3\phi_1$		$\cos 3\phi_1$	
3	9	$2/9\pi$	$1/\sqrt{2}$	$-2/3\pi$	$2/25\pi$	$\sin 3\phi_1$		$\cos 3\phi_1$	
5	7	$-2/7\pi$	$-1/\sqrt{2}$	$2/5\pi$	$2/9\pi$	$\sin 5\phi_1$		$\cos 5\phi_1$	
5	9	$2/9\pi$	$1/\sqrt{2}$	$2/5\pi$	$2/17\pi$	$\sin 5\phi_1$		$\cos 5\phi_1$	

Appendix A (contd.)

$$h_6(t) = \frac{\sqrt{2}A}{\pi^2} \cos w_c t \cos \frac{w_m t}{4} \left[\sin\phi_1 \left(-\frac{2}{27\pi} - \frac{2}{85\pi} + \frac{2}{175\pi} \cdots \right) \right.$$

$$+ \sin 3\phi_1 \left(\frac{2}{135\pi} - \frac{2}{357\pi} - \frac{2}{675\pi} \cdots \right)$$

$$\left. + \sin 5\phi_1 \left(\frac{2}{275\pi} + \frac{2}{725\pi} \cdots \right) \cdots \right]$$

$$+ \frac{\sqrt{2}A}{\pi^2} \cos w_c t \sin \frac{w_m t}{4} \left[\cos\phi_1 \left(-\frac{2}{27\pi} - \frac{2}{85\pi} + \frac{2}{175\pi} \cdots \right) \right.$$

$$+ \cos 3\phi_1 \left(\frac{2}{135\pi} - \frac{2}{357\pi} - \frac{2}{675\pi} \cdots \right)$$

$$\left. + \cos 5\phi_1 \left(\frac{2}{275\pi} + \frac{2}{725\pi} \cdots \right) \cdots \right]$$

Case 5: $q = 4(m-n)-1;\ m>n$

Let these particular terms of $h_2(t)$ be given by $h_7'(t)$ and evaluate for the first few values of m and n.

$$h_7'(t) = A \cos w_c t \sum_n \sum_m \rho_n\ \alpha_n\ \beta_m\ \delta_{4(m-n)-1} \cos \frac{4(m-n)-1}{4} w_m t$$

$$\cdot \left[\cos(m-n) w_m t \sin m\phi_1 + \sin(m-n) w_m t \cos m\phi_1 \right]$$

Appendix A (contd.)

TABLE A-7

EVALUATION OF SERIES TERMS OF $h_7'(t)$

m	n	ρ_n	α_n	β_m	$\delta_{4(m-n)-1}$	$\sin m\phi_1$	Difference Component Of $\cos\frac{4(m-n)-1}{4}w_m t\ \cos(m-n)w_m t$	$\cos m\phi_1$	Difference Component Of $\cos\frac{4(m-n)-1}{4}w_m t\ \sin(m-n)w_m t$
3	1	$2/\pi$	$1/\sqrt{2}$	$-2/3\pi$	$-2/7\pi$	$\sin 3\phi_1$		$\cos 3\phi_1$	
5	1	$2/\pi$	$1/\sqrt{2}$	$2/5\pi$	$-2/15\pi$	$\sin 5\phi_1$	$\frac{1}{2}\cos\frac{-w_m t}{4}$	$\cos 5\phi_1$	$-\frac{1}{2}\sin\frac{-w_m t}{4}$
5	3	$-2/3\pi$	$1/\sqrt{2}$	$2/5\pi$	$-2/7\pi$	$\sin 5\phi_1$		$\cos 5\phi_1$	

Appendix A (contd.)

$$h_7(t) = \frac{\sqrt{2}A}{\pi^2} \cos w_c t \cos \frac{w_m t}{4} \left[\sin 3\phi_1 \left(\frac{2}{21\pi} \right) \right.$$

$$+ \sin 5\phi_1 \left(- \frac{2}{75\pi} + \frac{2}{105\pi} \right) \cdots \Big]$$

$$+ \frac{\sqrt{2}A}{\pi^2} \cos w_c t \sin \frac{w_m t}{4} \left[\cos 3\phi_1 \left(\frac{2}{21\pi} \right) \right.$$

$$+ \cos 5\phi_1 \left(- \frac{2}{75\pi} + \frac{2}{105\pi} \right) \cdots \Big]$$

<u>Case 6: $q = 4(m-n)+1;\ m>n$</u>

Let these particular terms of $h_2(t)$ be given by $h_8'(t)$ and evaluate for the first few values of m and n.

$$h_8'(t) = A \cos w_c t \sum_n \sum_m \rho_n \alpha_n \beta_m \delta_{4(m-n)+1} \cos \frac{4(m-n)+1}{4} w_m t$$

$$\cdot \left[\cos(m-n) w_m t \sin m\phi_1 + \sin(m-n) w_m t \cos m\phi_1 \right]$$

Appendix A (contd.)

TABLE A-8

EVALUATION OF SERIES TERMS OF $h_8'(t)$

m	n	ρ_n	α_n	β_m	$\delta_{4(m-n)+1}$	Difference Component Of $\cos\frac{4(m-n)+1}{4}w_m t \cos(m-n)w_m t$: $\sin m\phi_1$		Difference Component Of $\cos\frac{4(m-n)+1}{4}w_m t \sin(m-n)w_m t$: $\cos m\phi_1$	
3	1	$2/\pi$	$1/\sqrt{2}$	$-2/3\pi$	$2/9\pi$	$\sin 3\phi_1$		$\cos 3\phi_1$	
5	1	$2/\pi$	$1/\sqrt{2}$	$2/5\pi$	$2/17\pi$	$\sin 5\phi_1$	$\frac{1}{2}\cos\frac{w_m t}{4}$	$\cos 5\phi_1$	$-\frac{1}{2}\sin\frac{w_m t}{4}$
5	3	$-2/3\pi$	$1/\sqrt{2}$	$2/5\pi$	$2/9\pi$	$\sin 5\phi_1$		$\cos 5\phi_1$	

Appendix A (contd.)

$$h_8(t) = \frac{\sqrt{2}A}{\pi^2} \cos w_c t \cos \frac{w_m t}{4} \left[\sin 3\phi_1 \left(- \frac{2}{27\pi} \right) \right.$$

$$+ \sin 5\phi_1 \left(\frac{2}{85\pi} - \frac{2}{135\pi} \right) \cdots \Big]$$

$$+ \frac{\sqrt{2}A}{\pi^2} \cos w_c t \sin \frac{w_m t}{4} \left[\cos 3\phi_1 \left(\frac{2}{27\pi} \right) \right.$$

$$+ \cos 5\phi_1 \left(- \frac{2}{85\pi} + \frac{2}{135\pi} \right) \cdots \Big]$$

Case 7: $q = 1$; $m = n$

Let these particular terms of $h_2(t)$ be given by $h_9(t)$ and evaluate for first few values of m and n.

$$h_9(t) = A \cos w_c t \sum_n \sum_m \rho_n \ \alpha_n \ \beta_m \ \delta_1 \cos \frac{w_m t}{4} \sin m\phi_1$$

TABLE A-9

EVALUATION OF SERIES TERMS OF $h_9(t)$

m	n	ρ_n	α_n	β_m	δ_1	$\sin m\phi_1$
1	1	$2/\pi$	$1/\sqrt{2}$	$2/\pi$	$2/\pi$	$\sin\phi_1$
3	3	$-2/3\pi$	$1/\sqrt{2}$	$-2/3\pi$	$2/\pi$	$\sin 3\phi_1$
5	5	$2/5\pi$	$-1/\sqrt{2}$	$2/5\pi$	$2/\pi$	$\sin 5\phi_1$

Appendix A (contd.)

$$h_9(t) = \frac{\sqrt{2}A}{\pi^2} \cos w_c t \cos \frac{w_m t}{4} \left[\frac{4}{\pi} \sin\phi_1 + \frac{4}{9\pi} \sin 3\phi_1 - \frac{4}{25\pi} \sin 5\phi_1 \cdots \right]$$

6.6 Carrier Terms From Fourth Term of $f_4(t)$

When $m = n, h_1(t)$ produces carrier terms that pass the filter. Also for certain values of q, i.e., $q = 4(n+m)$, over all m n; $q = 4(n-m)$, $n>m$; $q = 4(m-n)$, $m>n$, carrier terms will be produced by $h_1(t)$. Let the sum of these terms be denoted by $h_{10}(t)$.

$$h_{10}(t) = C_o \cos w_c t$$

6.7 Crystal Filter Output Signal

The combined signal components at the output of the crystal filter are given by $f_5(t)$.

$$f_5(t) = g_o(t) + g_4(t) + g_6(t) + h_3(t) + h_4(t) + h_5(t) + h_6(t) + h_7(t) + h_8(t) + h_9(t) + h_{10}(t)$$

Let $K = \frac{\sqrt{2}A}{\pi^2}$ and $g_o(t) + h_{10}(t) = C \cos w_c t$, then

$$f_5(t) = C \cos w_c t + 0.471K \sin \frac{w_m t}{4} \cos w_c t + \frac{2K}{\pi} \sin \frac{w_m t}{4} \cos w_c t \left[0.097 \cos\phi_1 + 0.064 \cos 3\phi_1 + 0.013 \cos 5\phi_1 \cdots\right] + \frac{2K}{\pi} \cos \frac{w_m t}{4} \cos w_c t \left[2.040 \sin\phi_1 + 0.229 \sin 3\phi_1 - 0.077 \sin 5\phi_1 \cdots\right]$$

Appendix A (contd.)

$$f_5(t) = C \cos w_c t + 0.471\ K \sin \frac{w_m t}{4} \cos w_c t$$
$$+ 0.062\ K \sin \frac{w_m t}{4} \cos w_c t \left[\cos\phi_1 + 0.660 \cos 3\phi_1\right.$$
$$\left. + 0.134 \cos 5\phi_1 \cdots\right] + 1.299\ K \cos \frac{w_m t}{4} \cos w_c t \left[\sin\phi_1\right.$$
$$\left. + 0.112 \sin 3\phi_1 - 0.038 \sin 5\phi_1 \cdots\right]$$

7.0 SIGNAL AT THE OUTPUT OF IF AMPLIFIER

The IF amplifier is assumed to have a constant gain, G, (AGC is disconnected). The gain factor takes into account any losses in gate, crystal filter, etc. Let

$$A_1 = GA \quad , \quad K_1 = GK \quad , \quad \text{and} \quad C_1 = GC$$

$$f_6(t) = C_1 \cos w_c t + 0.471\ K_1 \sin \frac{w_m t}{4} \cos w_c t$$
$$+ 0.062\ K_1 \sin \frac{w_m t}{4} \cos w_c t \left[\cos\phi_1 + 0.660 \cos 3\phi_1\right.$$
$$\left. + 0.134 \cos 5\phi_1 \cdots\right] + 1.299\ K_1 \cos \frac{w_m t}{4} \cos w_c t \left[\sin\phi_1\right.$$
$$\left. + 0.112 \sin 3\phi_1 - 0.038 \sin 5\phi_1 \cdots\right]$$

8.0 SIGNAL AT OUTPUT OF ENVELOPE DETECTOR

The envelope detector removes the $\cos w_c t$ term from $f_6(t)$. Thus the output of the envelope detector is given by:

$$f_7(t) = C_1 + 0.471\ K_1 \sin \frac{w_m t}{4} + 0.062\ K_1 \sin \frac{w_m t}{4} \left(\cos\phi_1\right.$$
$$\left. + 0.660 \cos 3\phi_1 + 0.134 \cos 5\phi_1 \cdots\right)$$
$$+ 1.299\ K_1 \cos \frac{w_m t}{4} \left(\sin\phi_1 + 0.112 \sin 3\phi_1 - 0.038 \sin 5\phi_1 \cdots\right)$$

Appendix A (contd.)

It is seen that the output of the envelope detector (input to the synchronous detector) consists of a DC term, and an in-phase and quadrature term with respect to the reference input $\cos \frac{w_m t}{4}$ signal at the other input port of the synchronous detector.

Thus,

$$f_{REF}(t) = \cos \frac{w_m t}{4}$$

$$f_7(t) = f_i(t) + f_q(t) + \text{DC term}$$

where:

$$f_i(t) = 1.299\ K_1 \cos \frac{w_m t}{4}\left(\sin\phi_1 + 0.112 \sin 3\phi_1 - 0.038 \sin 5\phi_1 \ldots\right)$$

$$f_q(t) = 0.471\ K_1 \sin \frac{w_m t}{4} + 0.062\ K_1 \sin \frac{w_m t}{4}\left(\cos\phi_1\right.$$

$$\left. + 0.660 \cos 3\phi_1 + 0.134 \cos 5\phi_1 \cdots\right)$$

9.0 SIGNAL AT OUTPUT OF SYNCHRONOUS DETECTOR

The signal at the output of the synchronous detector is given by:

$$f_8(t) = f_7(t) \cdot f_{REF}(t)$$

$$\therefore f_8(t) = C_1 \cos \frac{w_m t}{4} + 0.471\ K_1 \sin \frac{w_m t}{4} \cos \frac{w_m t}{4}$$

$$+ 0.062\ K_1 \sin \frac{w_m t}{4} \cos \frac{w_m t}{4}\left(\cos\phi_1 + 0.660 \cos 3\phi_1\right.$$

$$\left. + 0.134 \cos 5\phi_1 \cdots\right) + 1.299\ K_1 \cos \frac{w_m t}{4} \cos \frac{w_m t}{4}\left(\sin\phi_1\right.$$

$$\left. + 0.112 \sin 3\phi_1 - 0.038 \sin 5\phi_1 \cdots\right)$$

Appendix A (contd.)

$$f_8(t) = C_1 \cos \frac{w_m t}{4} + 0.471\ K_1\ \frac{1}{2}\left[\sin \frac{2w_m t}{4} + \sin o\right]$$

$$+ 0.062\ K_1\ \frac{1}{2}\left[\sin\left(\frac{2w_m t}{4}\right) + \sin(o)\right]\left(\cos\phi_1 + 0.660 \cos 3\phi_1\right.$$

$$\left. + 0.134 \cos 5\phi_1 \cdots\right) + 1.299\ K_1\ \frac{1}{2}\left[\cos\left(\frac{2w_m t}{4}\right) + \cos(o)\right]$$

$$\left(\sin\phi_1 + 0.112 \sin 3\phi_1 - 0.038 \sin 5\phi_1 \cdots\right)$$

10.0 ERROR SIGNAL OUTPUT FROM LOW PASS FILTER

Only the "DC" component of $f_8(t)$ will pass the filter. Note that this component is generated from the in-phase term, $f_i(t)$, in the IF signal. The error signal at the output of the filter (input to the VCXO) is now simply a function of the tracking phase error, ϕ_1.

$$V(\phi_1) = 0.650\ K_1\left(\sin\phi_1 + 0.112 \sin 3\phi_1 - 0.038 \sin 5\phi_1 \cdots\right)$$

or equivalently,

$$V(\phi_1) = 0.093\ A_1\left(\sin\phi_1 + 0.112 \sin 3\phi_1 - 0.038 \sin 5\phi_1 \cdots\right)$$

This equation represents the error signal (with no voice modulation). It is graphed in Figure 4 of the main text. It is important to note that the equation is truncated and therefore the resultant graph will closely approximate the exact solution.

APPENDIX B

ERROR SIGNAL FOR LM FINE TONE TRACKING LOOP
(WITH VOICE MODULATION)

1.0 INTRODUCTION

When voice modulation is present, the received ranging signal at the input to the LM receiver consists of an intermittent series of fine tone pulses amplitude modulated on the carrier. The error signal generated by the LM fine tone tracking loop under these signal conditions is derived in this Appendix.

A signal analysis, similar to the one performed in Appendix A, is presented here. The resultant error signal is shown to be usable for phase locking the locally generated fine tone (derived from the VCXO) to the received intermittent fine tone. Thus the LM fine tone tracking loop performs as desired.

2.0 FOURIER SERIES EXPANSION OF CLIPPED VOICE WAVEFORM

When the voice plus ranging mode is selected, the fine tone is gated on or off according to the polarity of the clipped voice waveform, see Figure 3d. The resultant intermittent series of fine tone pulses is then used to gate the carrier ON-OFF.

The clipped voice waveform is assumed to be derived from an audio (voice) signal which has frequencies restricted to the band of 300-3000 Hz. In order to simplify the analysis here, the clipped voice waveform first is assumed to be a rectangular waveform with a repetition lying in this frequency band. Results are then generalized for a clipped voice waveform having random pulse widths.

The Fourier series expansion for the clipped voice waveform (ON-OFF) is derived in the fashion outlined in Appendix A, paragraph 2.0. Thus the clipped voice ON-OFF gate is given by:

$$f_o(t) = \frac{1}{2}\left[1+2\sum_{r=1}^{\infty}\frac{\sin\frac{r\pi}{2}}{\frac{r\pi}{2}}\cos\left(rw_V t + r\phi_V\right)\right].$$

where:

w_v = fundamental frequency of clipped voice waveform (radians per sec.)

$\phi_v = w_v \tau_1$ = arbitrary phase shift of waveform with respect to t=o, see Figure 3d.

3.0 ON-OFF SWITCHING OF IF CARRIER AT INPUT TO THE GATE

The received waveform at the input to the gate is given by $f_{2v}(t)$ where the "v" subscript hereafter indicates voice modulation present. Thus,

$$f_{2v}(t)=f_o(t)\cdot f_2(t)$$

$$f_{2v}(t) = \frac{A}{4} \cos w_c t \left[1+2 \sum_{r=1}^{\infty} \frac{\sin \frac{r\pi}{2}}{\frac{r\pi}{2}} \cos\left(rw_v t + r\phi_v\right)\right]\left[1+2 \sum_{m=1}^{\infty} \frac{\sin \frac{m\pi}{2}}{\frac{m\pi}{2}} \cdot \cos\left(mw_m t + m\phi_1\right)\right]$$

4.0 THE GATING SIGNAL

The gating signal is unchanged in the voice plus ranging mode. From Appendix A,

$$f_3(t) = \frac{1}{2}\left[1+2 \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{\frac{n\pi}{2}} \cos\left(nw_m t + n\phi_2(t)\right)\right]$$

5.0 SIGNAL AT INPUT TO THE CRYSTAL FILTER

The signal into the crystal filter (output of the gate) is given by $f_{4v}(t) = f_{2v}(t).f_3(t)$

$$f_{4v}(t) = \frac{A}{8} \cos w_c t \left[1+2 \sum_m \beta_m \cos\left(mw_m t + m\phi_1\right)\right]\left[1+2 \sum_r \lambda_r \cos\left(rw_v t + r\phi_v\right)\right] \cdot \left[1+2 \sum_n \rho_n \cos\left(nw_m t + n\phi_2(t)\right)\right]$$

Appendix B (contd.)

with $w_v << w_m$

and $\lambda_r = \dfrac{\sin \frac{r\pi}{2}}{r\pi/2}$.

$$f_{4v}(t) = \frac{A}{8} \cos w_c t \Big[1 + 2 \sum_m \beta_m \cos\left(m w_m t + m\phi_1\right) + 2 \sum_r \lambda_r \cos\left(r w_v t + r\phi_v\right) + 4 \sum_m \sum_r \beta_m \lambda_r \cos\left(m w_m t + m\phi_1\right) \cos\left(r w_v t + r\phi_v\right) \Big] \Big[1 + 2 \sum_n \rho_n \cos\left(n w_m t + n\phi_2\right) \Big]$$

Thus the error signal at the input to the crystal filter is given by:

$$\begin{aligned} f_{4v}(t) = \frac{A}{8} \cos w_c t \Big[& 1 + 2 \sum_m \beta_m \cos\left(m w_m t + m\phi_1\right) + 2 \sum_r \lambda_r \cos\left(r w_v t + r\phi_v\right) + 4 \sum_m \sum_r \beta_m \lambda_r \cos\left(m w_m t + m\phi_1\right) \cos\left(r w_v t + r\phi_v\right) \\ & + 2 \sum_n \rho_n \cos\left(n w_m t + n\phi_2\right) + 4 \sum_m \sum_n \beta_m \rho_n \cos\left(m w_m t + m\phi_1\right) \cos\left(n w_m t + n\phi_2\right) \\ & + 4 \sum_r \sum_n \lambda_r \rho_n \cos\left(r w_v t + r\phi_v\right) \cos\left(n w_m t + n\phi_2\right) \\ & + 8 \sum_m \sum_r \sum_n \beta_m \lambda_r \rho_n \cos\left(m w_m t + m\phi_1\right) \cos\left(r w_v t + r\phi_v\right) \cos\left(n w_m t + n\phi_2\right) \Big] \end{aligned}$$

6.0 SIGNAL AT OUTPUT OF THE CRYSTAL FILTER

6.1 General

Each of the terms of $f_{4v}(t)$ are examined here to determine which components pass the filter.

Appendix B (contd.)

6.2 First Term Of $f_{4v}(t)$

$$a_1(t) = \frac{A}{8} \cos w_c t = \frac{1}{2} \left[g_o(t)\right]$$

This term represents an IF carrier component which passes the filter.

6.3 Sidebands From Second Term Of $f_{4v}(t)$

$$a_2(t) = \frac{A}{4} \cos w_c t \sum_{m=1}^{\infty} \beta_m \cos\left(mw_m t + m\phi_1\right)$$

This signal has sidebands at $w_c \pm mw_m$ and thus none of these sidebands can pass the filter.

6.4 Sidebands From Third Term Of $f_{4v}(t)$

$$a_3(t) = \frac{A}{4} \cos w_c t \sum_r \lambda_r \cos\left(rw_v t + r\phi_v\right)$$

$$a_3(t) = g_o(t) \sum_r \lambda_r \cos\left(rw_v t + r\phi_v\right)$$

This term produces sidebands around w_c at $\pm rw_v$, $r = 1, 2 \cdots$. The sidebands of significant amplitude will pass the filter.

6.5 Sidebands From Fourth Term Of $f_{4v}(t)$

$$a_4(t) = \frac{A}{2} \cos w_c t \sum_m \sum_r \beta_m \lambda_r \cos\left(mw_m t + m\phi_1\right) \cos\left(rw_v t + r\phi_v\right)$$

This term produces sidebands at $w_c \pm mw_m \pm rw_v$. For $m = 1$ and $w_v = 2\pi(3000)$, a minimum value of $r = 13$ must be used to obtain a sideband which passes the filter. These sidebands $(m \geq 13)$ have much smaller amplitudes than the sidebands of the third term of $f_{4v}(t)$, see above, and thus this term will be ignored.

6.6 Sidebands From Fifth Term Of $f_{4v}(t)$

$$a_5(t) = \frac{A}{4} \cos w_c t \sum_n \rho_n \cos\left[nw_m t + n\phi_2\right]$$

The sidebands of this function which pass the filter have been determined in Appendix A. Thus,

$$a_5(t) = \frac{1}{2}\left[g_4(t) + g_6(t)\right]$$

6.7 Sidebands From Sixth Term Of $f_{4v}(t)$

$$a_6(t) = \frac{A}{2} \cos w_c t \sum_n \sum_m \rho_n \beta_m \cos\left[nw_m t + n\phi_2\right]\cos\left[mw_m t + m\phi_1\right]$$

The sidebands of this function which pass the filter have been determined in Appendix A.

$$a_6(t) = \frac{1}{2}\left[h_3(t) + h_4(t) + h_5(t) + h_6(t) + h_7(t) + h_8(t) + h_9(t) + h_{10}(t)\right]$$

6.8 Sidebands From Seventh Term Of $f_{4v}(t)$

$$a_7(t) = \frac{A}{2} \cos w_c t \sum_n \rho_n \cos\left[nw_m t + n\phi_2\right] \cdot \left[\sum_r \lambda_r \cos\left[rw_v t + r\phi_v\right]\right]$$

From Appendix A it is seen that:

$$a_7(t) = \left[g_4(t) + g_6(t)\right] \sum_r \lambda_r \cos\left[rw_v t + r\phi_v\right]$$

Sidebands from $a_7(t)$ occur at $w_c \pm \frac{w_m}{4} \pm rw_v$. The majority of these sidebands pass the filter.

Appendix B (contd.)

6.9 Sidebands From Eighth Term Of $f_{4v}(t)$

$$a_8(t) = A \cos w_c t \sum_n \sum_m \rho_n \beta_m \cos\left[n w_m t + n\phi_2\right]$$

$$\cdot \cos\left[m w_m t + m\phi_1\right]\left[\sum_r \lambda_r \cos\left[r w_v t + r\phi_v\right]\right]$$

From Appendix A it is seen that:

$$a_8(t) = \left[h_3(t) + h_4(t) + h_5(t) + h_6(t) + h_7(t) + h_8(t) + h_9(t) + h_{10}(t)\right]\sum_r \lambda_r \cos\left[r w_v t + r\phi_v\right]$$

Sidebands from $a_8(t)$ occur at $w_c \pm \frac{w_m}{4} \pm r w_v$. The majority of these sidebands pass the filter.

6.10 Crystal Filter Output Signal

The combined signal components at the output of the crystal filter are given by $f_{5v}(t)$.

$$f_{5v}(t) = a_1(t) + a_3(t) + a_5(t) + a_6(t) + a_7(t) + a_8(t)$$

$$f_{5v}(t) = \left\{\left[g_o(t) + h_{10}(t)\right] + \left[g_4(t) + g_6(t)\right] + \left[h_3(t) + h_4(t) + h_5(t) + h_6(t) + h_7(t) + h_8(t) + h_9(t)\right]\right\}\left[\frac{1}{2} + \sum_r \lambda_r \cos\left[r w_v t + r\phi_v\right]\right]$$

Appendix B (contd.)

$$f_{5v}(t) = \frac{C}{2}\cos w_c t + C\cos w_c t \sum_r \lambda_r \cos\left(rw_v t + r\phi_v\right)$$

$$+ 0.236\ K\cos w_c t \sin\frac{w_m t}{4}$$

$$+ 0.471\ K\cos w_c t \sin\frac{w_m t}{4}\sum_r \lambda_r \cos\left(rw_v t + r\phi_v\right)$$

$$+ 0.031\ K\cos w_c t \sin\frac{w_m t}{4}\left(\cos\phi_1 + 0.660\cos 3\phi_1\right.$$

$$\left.+ 0.134\cos 5\phi_1 \cdots\right) + 0.650\ K\cos w_c t\cos\frac{w_m t}{4}\left(\sin\phi_1\right.$$

$$\left.+ 0.112\sin 3\phi_1 - 0.038\sin 5\phi_1 \cdots\right)$$

$$+ \left\{0.062\ K\cos w_c t\sin\frac{w_m t}{4}\left(\cos\phi_1 + 0.660\cos 3\phi_1\right.\right.$$

$$\left.+ 0.134\cos 5\phi_1 \cdots\right) + 1.299\ K\cos w_c t\cos\frac{w_m t}{4}\left(\sin\phi_1\right.$$

$$\left.\left.+ 0.112\sin 3\phi_1 - 0.038\sin 5\phi_1 \cdots\right)\right\}$$

$$\cdot \sum_r \lambda_r \cos\left(rw_v t + r\phi_v\right)$$

7.0 SIGNAL AT OUTPUT OF SYNCHRONOUS DETECTOR

As shown in Figure 2, $f_{5v}(t)$ is amplified and envelope detected; the resultant signal is multiplied by a coherent $\cos\frac{w_m t}{4}$ at the synchronous detector. The steps involved are outlined in detail in Appendix A. For convenience these intermediate steps are omitted here. The output of the synchronous detector is given by:

Appendix B (contd.)

$$
\begin{aligned}
f_{8v}(t) = {} & \frac{C_1}{2} \cos \frac{w_m t}{4} + C_1 \cos \frac{w_m t}{4} \sum_r \lambda_r \cos\left(r w_v t + r\phi_v\right) \\
& + \frac{1}{2}(0.236) K_1 \sin \frac{2w_m t}{4} \\
& + \frac{1}{2}(0.031) K_1 \sin \frac{2w_m t}{4}\left(\cos\phi_1 + 0.660 \cos 3\phi_1\right. \\
& \left. + 0.134 \cos 5\phi_1 \cdots\right) + \frac{1}{2}(0.650) K_1 \cos(o)\left(\sin\phi_1\right. \\
& \left. + 0.112 \sin 3\phi_1 - 0.038 \sin 5\phi_1 \cdots\right) \\
& + \frac{1}{2}(0.650) K_1 \cos \frac{2w_m t}{4}\left(\sin\phi_1 + 0.112 \sin 3\phi_1\right. \\
& \left. - 0.038 \sin 5\phi_1 \cdots\right) + \frac{1}{2}(0.471) K_1 \sin \frac{2w_m t}{4} \sum_r \lambda_r \cos\left(r w_v t\right. \\
& \left. + r\phi_v\right) + \frac{1}{2}(0.062) K_1 \sin \frac{2w_m t}{4}\left(\cos\phi_1 + 0.660 \cos 3\phi_1\right. \\
& \left. + 0.134 \cos 5\phi_1 \cdots\right) \cdot \sum_r \lambda_r \cos\left(r w_v t + r\phi_v\right) \\
& + \frac{1}{2}(1.299) K_1 \cos \frac{2w_m t}{4}\left(\sin\phi_1 + 0.112 \sin 3\phi_1\right. \\
& \left. - 0.038 \sin 5\phi_1 \cdots\right) \cdot \sum_r \lambda_r \cos\left(r w_v t + r\phi_v\right) \\
& + \frac{1}{2}(1.299) K_1 \cos(o)\left(\sin\phi_1 + 0.112 \sin 3\phi_1\right. \\
& \left. - 0.038 \sin 5\phi_1 \cdots\right) \cdot \sum_r \lambda_r \cos\left(r w_v t + r\phi_v\right)
\end{aligned}
$$

Appendix B (contd.)

8.0 ERROR SIGNAL OUTPUT FROM LOW PASS FILTER

Only the "DC" component from $f_{8v}(t)$ will pass the low pass filter. Note that sidebands from the $\sum_r \lambda_r \cos\left(rw_v t + r\phi_v\right)$ terms fall outside the filter bandpass and thus no contribution can be received from terms of $f_{8v}(t)$ which contain this function. This will be true even if a random pulse width clipped voice waveform were to be used in the analysis.

Thus the only term of $f_{8v}(t)$ which will pass the low pass filter is the fifth term.

$$V_v(\phi_1) = 0.325\ K_1\left(\sin\phi_1 + 0.112\ \sin 3\phi_1 - 0.038\ \sin 5\phi_1 \cdots\right)$$

or equivalently,

$$V_v(\phi_1) = 0.047\ A_1\left(\sin\phi_1 + 0.112\ \sin 3\phi_1 - 0.038\ \sin 5\phi_1 \cdots\right)$$

Note that $V_v(\phi_1) = \frac{1}{2}\,V(\phi_1)$ i.e., the error signal amplitude (and slope at crossover) is reduced by a factor of two when voice modulation is present.

This function is also plotted in Figure 4.